Part I: The $T\bar{T}$ deformation

Lecture 1

In this lecture we motivate the study of the $T\bar{T}$ deformation, introduce the $T\bar{T}$ operator and describe some of its properties.

1 Introduction

This is a class on holography and the $T\bar{T}$ deformation. Our goal will be to study models of quantum gravity (QG) by deforming well-understood quantum field theories (QFTs) by the $T\bar{T}$ operator, a novel composite operator made up of components of the stress tensor. This operator is present in any Poincaré-invariant QFT satisfying a reasonable set of assumptions. In particular, we will see that the expectation value of the $T\bar{T}$ operator is finite, factorizes, and has a fixed conformal dimension. These properties allow us to define a special type of deformation where the theory is continuously deformed by the $T\bar{T}$ operator in a way that we'll describe shortly. Remarkably, the $T\bar{T}$ deformation can be understood as a model of quantum gravity in different dimensions. On the one hand, any $T\bar{T}$-deformed QFT is equivalent to a two-dimensional theory of flat Jackiw-Teitelboim (JT) gravity. On the other hand, the $T\bar{T}$ deformation of a holographic conformal field theory (CFT) can be interpreted as a theory of quantum gravity in three-dimensional spacetimes.

1.1 Motivation

Why are we interested in quantum gravity? One reason is that understanding quantum gravity is crucial to describe the origin of the universe and the emergence of spacetime. Another more fundamental reason is that gravity is special and unlike the three other forces of nature. While the electromagnetic, weak, and strong forces have been successfully described using QFT in what's known as the Standard Model of particle physics, the same is not true for quantum gravity. There are many reasons for this. One reason is that gravity is non-renormalizable in four dimensions. This follows from the fact that the gravitational coupling constant, G_N, is dimensionful with $[G_N] = L^2$. The Einstein-Hilbert action expanded around a fixed Minkowski background $\eta_{\mu\nu}$ via $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ is given by

$$I_{EH} = \frac{1}{16\pi G_N}\int \sqrt{-g}\, R d^4x \sim \frac{1}{16\pi G_N}\int d^4x \big(\partial h \partial h + \underbrace{G_N^{1/2} h\partial h \partial h + G_N h^2 \partial h \partial h + \dots}_{\text{non-renormalizable interactions}}\big), \tag{1.1}$$

where we have omitted all indices on the derivatives and the graviton, as well as all of the different contractions of these indices. This action describes a graviton h $(= h_{\mu\nu})$ with an infinite number of self-interactions dictated by the gauge (diffeomorphism) invariance of the Einstein-Hilbert action. There is no obstacle in quantizing the free theory and treating the interactions perturbatively. However, since amplitudes must be dimensionless quantities, for a scattering process with a characteristic energy E, the amplitude must scale as

$$\mathcal{A} \sim (E G_N^{1/2})^n \sim \left(\frac{E}{M_{pl}}\right)^n, \qquad \text{for some } n \geq 1. \tag{1.2}$$

We see that there's a violation of perturbative unitarity when $E \gtrsim M_{pl} \sim 10^{18}$ GeV since the square of the amplitude cannot be greater than 1. The Planck mass M_{pl} is the scale where our effective description of quantum gravity as gravitons propagating on a fixed Minkowski background breaks down. We say that the theory described by (1.1) is not UV complete, meaning that it's not defined at arbitrarily high energies/short distances.

In the standard framework of QFT, a non-renormalizable theory that breaks down at a scale M_Λ must be replaced by another theory that features additional degrees of freedom and is valid at energies $E \geq M_\Lambda$. A classic example of this is the Fermi theory for the β decay of a neutron into a proton, an electron, and an electron neutrino ($n \to p + e + \bar{\nu}_e$). The interaction term in the Fermi theory is given by

$$I_{int}^{\text{Fermi}} = \int d^4x \, G_F \bar{\psi}_p \gamma^\mu \psi_n \bar{\psi}_{\nu_e} \gamma^\nu \psi_e, \tag{1.3}$$

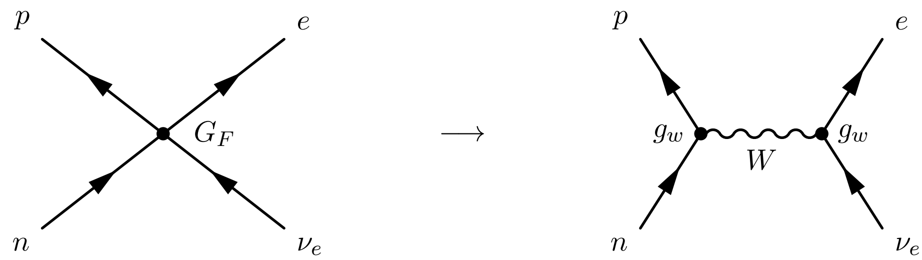
where $[G_F] = L^2$. The Fermi theory is valid for scales $E \ll G_F^{-1/2} \sim 100$ GeV. At this energy, Fermi's effective field theory description of β decay is replaced by a theory with additional degrees of freedom, the W bosons, with renormalizable interactions. These interactions are schematically given by

$$I_{int}^{\text{SM}} = \int d^4x \sum_{i,j} g_w \bar{\psi}_i \gamma^\mu W_\mu \psi_i, \tag{1.4}$$

where g_w is a dimensionless coupling related to G_F and the mass M_W of the W boson by

$$G_F \sim \frac{g_w^2}{M_W^2}. \tag{1.5}$$

The new theory extends the regime of validity of Fermi's theory to energies beyond $G_F^{-1/2}$. In terms of Feynman diagrams, the UV completion of Fermi's theory can be understood as



(1.6)

There is clearly an analogy between G_N and G_F (they even share the same dimensions!) However, the story is dramatically different in gravity because of the existence of black holes. This is another reason why gravity is special. In four dimensions, the simplest black hole is described by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{r_g}{r}\right)dt^2 + \left(1 - \frac{r_g}{r}\right)^{-1} dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{1.7}$$

where $r_g = 2MG_N$ is the location of the horizon. The Compton wavelength of a particle with energy E scales as $\lambda \sim 1/E$. Thus, at very high energies of order M_{pl}, the characteristic size of a particle is of the same order of magnitude as its Schwarzschild radius

$$\lambda \sim \frac{1}{M_{pl}}. \tag{1.8}$$

In other words, the scale where (1.1) breaks down is also the scale where any particle collapses into a black hole. This means that gravity doesn't fit within the Wilsonian paradigm at work in the Fermi theory (and the Standard Model). Instead, gravity has a minimum length scale, $\ell_p \sim M_p^{-1}$, and it's not clear what it means to do physics beyond this scale in the QFT framework.

In this class we will consider a simple two-dimensional model of quantum gravity that shares many features with four-dimensional gravity including non-normalizable interactions, a minimum length scale, and features characteristic of black holes. This theory does not fit within the standard framework of a local QFT and can be defined at arbitrarily small scales. The theory is obtained from a deformation of a free scalar field by the $T\bar{T}$ operator that we will introduce shortly. In fact, we will see that any $T\bar{T}$-deformed QFT can be interpreted as a theory of quantum gravity in two dimensions.

As mentioned above, black holes are another reason why gravity is special. In fact, the entropy of black holes tells us that a non-perturbative description of gravity, in particular one that is capable of describing black holes, requires a drastic departure from local QFT. In order to see this we note that the entropy S_R of a spatial subregion R in a d-dimensional QFT at finite temperature is given, using standard thermodynamic arguments, by

$$S_R \sim \text{Vol}(R)T^{d-1}, \tag{1.9}$$

where $\text{Vol}(R)$ is the volume of R and T is the temperature of the QFT. In contrast, the entropy of black holes S_{BH} scales like the area of the horizon

$$S_{BH} = \frac{\text{Area}}{4G}. \tag{1.10}$$

This calculation reveals a profound difference between gravity and quantum field theory. In fact, the entropy of any region in gravity is bounded from above by the entropy of a black hole in that region such that

$$S_R \leq S_{BH} = \frac{\text{Area}(R)}{4G}. \tag{1.11}$$

The fact that the entropy of black holes scales like the area is a manifestation of holography: the idea that quantum gravity in $(d+1)$-dimensions can be described by a lower dimensional theory without gravity. The most successful realization of this idea originated in string theory and is known as the AdS/CFT correspondence. The latter states that quantum gravity on $(d+1)$ asymptotically anti-de Sitter (AdS) spacetimes is described by a d-dimensional CFT "living" at the conformal boundary (see figure 1).

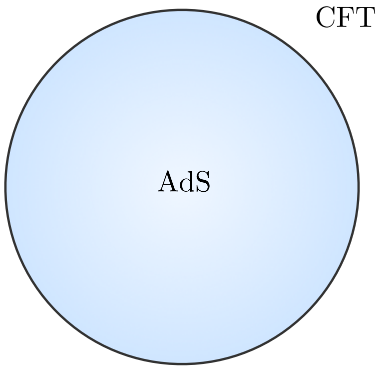


Figure 1: A cartoon of AdS/CFT where the disk represents gravity on an asymptotically AdS spacetime and the dual CFT can be thought of as of living at its conformal boundary.

The most studied example of the AdS/CFT correspondence is the duality between type IIB string theory on $\text{AdS}_5 \times S^5$ spacetimes and an $\mathcal{N}=4$ supersymmetric Yang-Mills theory in four dimensions. A more recent example of AdS/CFT which we will discuss in more detail in this class is the duality between type IIB string theory on $\text{AdS}_3 \times S^3 \times T^4$ and a deformation of the symmetric product orbifold of T^4.

Both string theory and the AdS/CFT correspondence provide us with a nonperturbative formulation of quantum gravity. In particular, AdS/CFT has led to many insights in quantum gravity including the counting of black hole microstates, the emergence of spacetime, and the fate of black hole evaporation. We expect these insights to be valid in other spacetimes beyond AdS. This motivates the search for holographic dualities capable of describing gravity in other spacetimes.

The $T\bar{T}$ deformation is also useful in this context. On the one hand, we will see that the $T\bar{T}$ deformation allows us to study three-dimensional gravity in locally AdS_3 spactimes satisfying a new type of boundary condition at the asymptotic boundary. For a particular sign of the $T\bar{T}$ deformation (negative in our conventions), these boundary conditions are equivalent to moving the asymptotic boundary of the spacetime into its interior; and this is the location where the $T\bar{T}$-deformed theory lives. This is illustrated in figure 2.[1]

[1]This equivalence holds for the *dobule-trace* version of the deformation and when the bulk theory of gravity

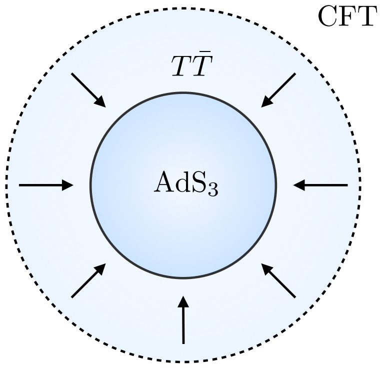


Figure 2: The $T\bar{T}$ deformation with a negative deformation parameter can be interpreted, in the context of holography, as moving the asymptotic boundary of locally AdS_3 spacetimes a finite distance into the bulk.

On the other hand, we will see that the $T\bar{T}$ deformation also allows us to construct new types of holographic dualities where the bulk spacetime is no longer AdS_3 (neither asymptotically nor locally), but a spacetime that looks nearly asymptotically flat. This model admits a formulation in string theory, which opens up the possibility of having an exact holographic duality for non-AdS spacetimes.

The class will be divided into three parts where we will describe the different ways in which the $T\bar{T}$ deformation can be used to construct models of quantum gravity. We will aim to cover the following topics:

I. The $T\bar{T}$ deformation and 2d gravity

- the $T\bar{T}$ operator, the spectrum, and torus partition function
- the $T\bar{T}$ deformation of a symmetric product orbifold and a free scalar.
- alternative formulations of $T\bar{T}$-deformed QFTs

II. Cutoff AdS_3 holography from double-trace $T\bar{T}$

- brief introduction to AdS_3/CFT_2
- cutoff AdS_3 holography and mixed boundary conditions
- holographic entanglement entropy

III. Non-AdS_3 holography from single-trace $T\bar{T}$

- brief introduction to string theory on AdS_3
- the $T\bar{T}$ deformation in string theory and its spectrum
- the $T\bar{T}$ deformation from supergravity, black hole entropy, and thermodynamics

is described by the Einstein-Hilbert action without matter fields.

1.2 The $T\bar{T}$ deformation

Let us consider a Poincaré-invariant QFT in two dimensions. The symmetry group is

$$ISO(1,1) = \underbrace{SO(1,1)}_{\text{Lorentz boosts}} \ltimes \underbrace{U(1)^2}_{\text{translations}} . \tag{1.12}$$

With the exception of supersymmetry and conformal symmetry, the Coleman-Mandula theorem tells us that this is the largest symmetry group of spacetime symmetries in any 2d QFT with a nontrivial S-matrix. In two dimensions, the Minkowski metric is simply given by

$$ds^2 = -dt^2 + dx^2. \tag{1.13}$$

In terms of these coordinates, the $ISO(1,1)$ transformations correspond to

$$\text{translations: } x^\mu \to x^\mu + \xi^\mu, \tag{1.14}$$

$$\text{boosts: } x^\mu \to x^\mu + \epsilon^{\mu\nu} x_\nu, \tag{1.15}$$

where $x^\mu \in (t,x)$, $\epsilon^{tx} = -\epsilon^{xt} = \beta$, and ξ^t, ξ^x, β are constants. These symmetries are generated by the stress tensor $T_{\mu\nu}$ of the theory via the Noether currents

$$\text{translations:} \quad j^\mu_\xi = T^\mu{}_\nu \xi^\nu \qquad \Longrightarrow \qquad \begin{array}{l} E = \int dx\, j^t_{\xi^t} = \int dx\, T^t{}_t \xi^t, \\ P = \int dx\, j^t_{\xi^x} = \int dx\, T^t{}_x \xi^x, \end{array} \tag{1.16}$$

$$\text{boosts:} \quad \tilde{j}^\mu_\beta = x^\alpha T^{\mu\beta} \epsilon_{\alpha\beta}, \qquad \Longrightarrow \qquad B = \int dx \big(t T^t{}_x + x T^t{}_t\big)\beta. \tag{1.17}$$

It's not difficult to verify that the Noether current generating translations in (1.16) is conserved as a consequence of the conservation of the stress tensor

$$\partial_\mu T^\mu{}_\nu. \tag{1.18}$$

Similarly, the Noether current generating Lorentz boosts in (1.17) is conserved provided that, in addition to (1.18), the stress tensor is symmetric such that

$$T_{\mu\nu} = T_{\nu\mu}. \tag{1.19}$$

Any local, Poincaré-invariant QFT in d-dimensions has a conserved and symmetric stress tensor. Hence, the stress tensor is a universal operator in this class of QFTs. Another operator that exists in any Poincaré-invariant QFT in two dimensions is the so-called $T\bar{T}$ operator. Let us introduce a new set of coordinates that will be convenient throughout this class. These are the so-called lightcone coordinates

$$x^\pm = x \pm t. \tag{1.20}$$

The name of these coordinates stems from the fact that massless particles propagate along fixed values of x^+ or x^-. In these coordinates, the components of the stress tensor can be written as $T_{\pm\pm}$ and $T_{\pm\mp}$. The $T\bar{T}$ operator is then defined as

$$T\bar{T} := \lim_{y^\pm \to x^\pm} \left[T_{++}(x^\pm) T_{--}(y^\pm) - T_{+-}(x^\pm) T_{-+}(y^\pm) \right]. \tag{1.21}$$

Exercise 1.1: write the conserved currents (1.16) and (1.17) in terms of $T_{\pm\pm}$ and $T_{\pm\mp}$. Verify the conservation of the currents. What hapens to the translation currents when $T_{\pm\mp} = 0$? Hint: allow ξ^μ to depend on coordinates.

Generically, the coincident limit of two operators is divergent in any QFT. However, we will see that in two dimensions, the $T\bar{T}$ operator is finite up to possibly divergent terms that don't contribute to its expectation value. Since the stress tensor has scaling dimension 2, the $T\bar{T}$ operator has scaling dimension 4, at least classically. In general, a composite operator acquires anomalous dimensions that depend on the details of the theory unless the operator is protected by symmetry. We will see that the $T\bar{T}$ operator is such an operator, meaning that it has fixed scaling dimension 4 in any Poincaré-invariant two-dimensional QFT satisfying a reasonable set of assumptions. These facts, which will be derived in detail the next lecture, lead us to the following conclusion:

The $T\bar{T}$ operator is a well-defined operator in any Poincaré-invariant quantum field theory in two-dimensions.

Let us now consider deforming a QFT by an operator $\mathcal{O}_\Delta$ with scaling dimension Δ. The infinitesimal change of the action I is then given by

$$\delta I = \lambda \int d^d x \, \mathcal{O}_\Delta, \tag{1.22}$$

where $\lambda \ll 1$ is the deformation parameter with dimension $[\lambda] = L^{\Delta - d}$. Depending on the value of Δ we have the following kinds of deformations (see figure 3):

(i) Relevant deformations ($\Delta < d$): these type of deformations induce an RG flow from the ultraviolet (UV) to the infrared (IR). In any local QFT there is a finite number of operators with $\Delta < d$, which means that the RG equations can be solved, in principle, given some initial data. For example, for a free scalar field in four dimensions, whose action is simply given by $I = \int d^4x \, \partial_\mu \phi \partial^\mu \phi$, the mass term

$$\delta I = m^2 \int d^4 x \, \phi^2, \tag{1.23}$$

is an example of a relevant deformation.

(ii) Marginal deformations ($\Delta = d$): when the QFT is a fixed point of the RG flow, the theory acquires, in addition to Poincaré invariance, scale invariance, as is described by a CFT. For certain CFTs with at least some internal symmetries or supersymmetries, a moduli space of deformations may exist where the theory remains conformal. The coordinates on this moduli space (or conformal manifold) correspond to the set $\{\mathcal{O}_d\}$ of marginal operators of the theory. For example, for a free compact scalar field in two dimensions, whose action can be written in lightcone coordinates as $I = \int d^2x\, \partial_+\phi\partial_-\phi$ there are two sets of conserved $U(1)$ currents, namely $j = \partial_+\phi$ and $\bar{j} = \partial_-\phi$, of scaling dimension 1. Thus, the deformation

$$\delta I = \lambda \int d^2x\, \mathcal{O}_2, \qquad \mathcal{O}_2 := j\bar{j}, \tag{1.24}$$

is (exactly) marginal and takes the theory to another point on its moduli space (it changes the radius of compactification). Marginal deformations of the current-current type (1.24) will play an important role in the third part of the class.

(*iii*) Irrelevant deformations ($\Delta > d$): this type of deformation is more difficult to deal with as it takes the theory up the RG flow, from the IR to the UV. Solving the RG equations requires, in principle, an infinite amount of data. One way to see this is that any QFT has an infinite number of operators with scaling dimension $\Delta > d$. We have already seen two examples of irrelevant interactions in four dimensions in this lecture, namely in the perturbative expansion of the Einstein-Hilbert action and in the Fermi theory. These irrelevant deformations take the form

$$\delta I_n = (G_N)^{n/2} \int d^4x\, h^n \partial h \partial h, \quad n > 0, \qquad \delta I = G_F \int d^4x\, \bar{\psi}\gamma^\mu\psi\bar{\psi}\gamma^\nu\psi, \tag{1.25}$$

where G_N and G_F have dimension L^2. Another example of an irrelevant deformation is the $T\bar{T}$ operator, which for a free scalar field in two dimensions, is given by

$$\delta I = \mu \int d^2x\, T\bar{T}, \qquad T\bar{T} = (\partial_+\phi)^2(\partial_-\phi)^2, \qquad [\mu] = L^2. \tag{1.26}$$

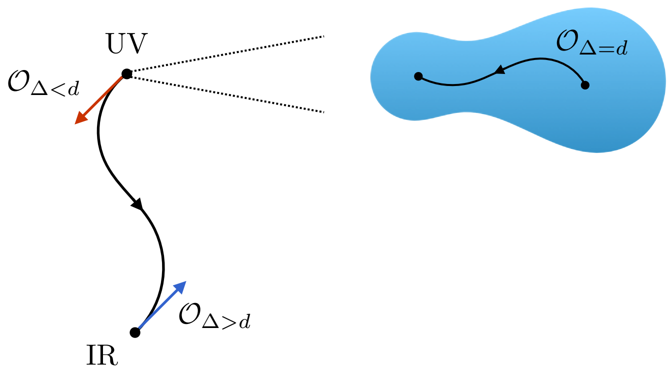


Figure 3: The effect of different types of deformations driven by the operator $\mathcal{O}_\Delta$.

We have mentioned that the $T\bar{T}$ operator is a well-defined operator with a fixed scaling dimension in any Poincaré-invariant QFT in two dimensions. This property of $T\bar{T}$ allows us to define a special kind of deformation where the original QFT is *continuously deformed* by the

$T\bar{T}$ operator at each step along the deformation, as illustrated in figure 4. The $T\bar{T}$ deformation is formally defined by the following differential equation for the action

$$\partial_\mu I(\mu) = 8\pi \int d^2x \, (T\bar{T})_\mu, \tag{1.27}$$

where $I(\mu)$ is the deformed action, $(T\bar{T})_\mu$ denotes the instantaneous $T\bar{T}$ operator, i.e. the operator of the deformed theory, and μ has dimensions of L^2. The differential equation (1.27) defines an infinite number of irrelevant interactions with fine-tuned coefficients determined by the $T\bar{T}$ operator, namely

$$I(\mu) = I(0) + 8\pi\mu \int d^2x \, (T\bar{T})_0 + \mathcal{O}(\mu^2). \tag{1.28}$$

It is important to note, however, that when we treat the deformation perturbatively as we have done above, we have to provide additional data, in the form of coefficients for other irrelevant operators (as is the case for *any* irrelevant deformation) that is not determined by (1.27). We will see this more explicitly later on when we consider the $T\bar{T}$ deformation of a free scalar field.

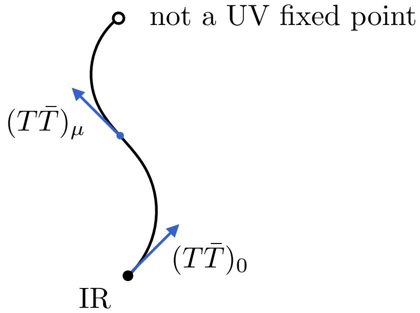


Figure 4: The $T\bar{T}$ deformation is an instantaneous deformation driven by $T\bar{T}$ operator at each point along the deformation.

In the introduction we motivated the $T\bar{T}$ deformation from the point of view of quantum gravity in two and three dimensions. However, the $T\bar{T}$ deformation is interesting in its own right and has some remarkable properties such as

(i) Universality: the $T\bar{T}$ operator is a well-defined operator in any Poincaré-invariant QFT in two dimensions. In particular, the $T\bar{T}$ deformation does not depend on the details of the undeformed QFT. Furthermore, Lorentz invariance turns out not to be strictly necessary so that the $T\bar{T}$ operator exists in an even larger class of theories invariant only under translations.

(ii) Solvability: the $T\bar{T}$ deformation is solvable in the sense that there is a universal differential equation for the spectrum and a universal expression for the S-matrix of the deformed theory. When the QFT is conformally invariant, i.e. when the QFT is actually a CFT, the spectrum can be solved explicitly and features a square-root structure.

(*iii*) UV completeness: a $T\bar{T}$-deformed QFT can be defined to arbitrarily high energies/short distances beyond the natural cutoff scale μ where one would expect new degrees of freedom (in the Wilsonian paradigm). Another way of saying this is that the $T\bar{T}$ deformation does not induce an RG flow or introduce new degrees of freedom. The price to pay for this is that the theory is non-local.

References

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[2] F. A. Smirnov and A. B. Zamolodchikov, *On space of integrable quantum field theories*, *Nucl. Phys.* **B915** (2017) 363 [1608.05499].

[3] A. Cavaglià, S. Negro, I. M. Szécsényi and R. Tateo, *$T\bar{T}$-deformed 2D Quantum Field Theories*, *JHEP* **10** (2016) 112 [1608.05534].