String theory on AdS_3

Let us consider string theory on $AdS_3 \times S^3 \times T^4$. We focus on the bosonic AdS_3 sector with NS-NS flux. The low energy effective description is IIB SUGRA:

$$I = \frac{2\pi}{(2\pi l_s)^8} \int d^{10}x \sqrt{|g|}\, e^{-2\phi} \left\{ R - 4\,\partial_\mu \phi\, \partial^\mu \phi - \frac{1}{12} H_{\mu\nu\alpha} H^{\mu\nu\alpha} \right\}$$

$\hookrightarrow H = dB$

The background features electric and magnetic charges

$$Q_e = \frac{1}{(2\pi l_s)^6} \int_{S^3 \times T^4} e^{-2\phi} * H = p, \qquad Q_m = \frac{1}{(2\pi l_s)^2} \int_{S^3} H = k$$

↳ F1 brane is electrically charged under B

↳ NS5 brane is magnetically charged under B

Related to the fact that $AdS_3 \times S^3 \times T^4$ originates from the near horizon limit of p F1 and k NS5 branes $\Rightarrow$ S-dual of the D1-D5 system.

The space of asymptotically AdS_3 solutions can be parametrized as

$$ds^2 = l^2 \left\{ \frac{dr^2}{4(r^2 - 4\bar{L}_u^2 \bar{L}_v^2)} - r\, du dv + \bar{L}_u^2\, du^2 + \bar{L}_v^2\, dv^2 \right\} + ds_{S^3} + ds_{T^4}$$

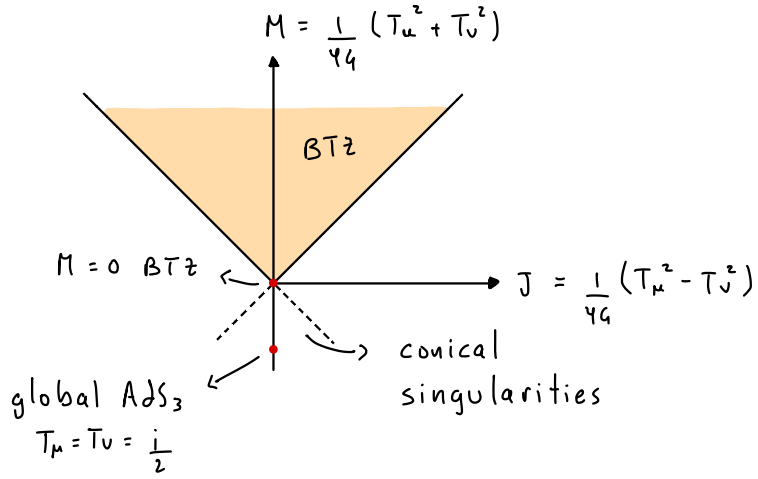
l: AdS scale

$u = t + \varphi$, $v = t - \varphi$, $\varphi \sim \varphi + 2\pi$

$$B = \frac{l^2}{2} r\, du \wedge dv + B_{\wedge_3}$$

$e^{2\phi} = \frac{k}{p}$, $k \equiv \frac{l^2}{l_s^2}$, $k \gg 1$, $p \gg 1$ ⟶ weak string coupling

↳ semiclassical limit



String theory on AdS_3 with NS-NS flux admits a perturbative worldsheet description as an $SL(2,\mathbb{R}) \times SU(2)$ WZW model,

$$S_{WZW} = \frac{k}{16\pi} \int_{\partial M} d^2z \, \mathrm{Tr}\left(\partial_a g^{-1} \partial^a g\right) + k\,\Gamma, \qquad g \in SL(2,\mathbb{R})$$

$$\Gamma \propto \int_M d^3x \, \epsilon^{abc} \, \mathrm{Tr}\left(g^{-1}\partial_a g \, \partial_b g^{-1} \partial_c g\right)$$

or as a NLSM. The AdS_3 part of the action is:

$$S = -\frac{1}{2l_s^2} \int d^2z \left(\sqrt{-h}\, h^{ab} G_{\mu\nu} + \epsilon^{ab} B_{\mu\nu}\right) \partial_a X^\mu(z,\bar z)\, \partial_b X^\nu(z,\bar z)$$

$\epsilon^{z\bar z} = -1$; target space coords: (ρ, u, v)

worldsheet coords $(z,\bar z) = (\tau+\sigma, \tau-\sigma)$

worldsheet metric $ds^2 = -dz\,d\bar z$

$$= \frac{1}{l_s^2} \int d^2z \, \Pi_{\mu\nu} \partial X^\mu \bar\partial X^\nu, \qquad \Pi_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$$

For global AdS_3:

$$S = k \int d^2z \left(\frac{\partial r \bar{\partial} r}{4r^2-1} - \frac{1}{4} \partial u \bar{\partial} u - \frac{1}{4} \partial v \bar{\partial} v - r \bar{\partial} u \partial v \right)$$

Exercise: Parametrize the group elements of $SL(2,\mathbb{R})$ by $T^3 = -\frac{i}{2}\sigma^2$, $T^{\pm} = \frac{1}{2}(\sigma^3 \pm i\sigma^1)$. Use $g = e^{uT^3} e^{\frac{1}{2}\log(2r + \sqrt{4r^2-1})(T^+ + T^-)} e^{vT^3}$ to show the EOM of the NLSM match those of the WZW model ($T_u = T_v = \frac{i}{2}$).

The worldsheet action is invariant under

$$\hat{SL}(2,\mathbb{R})_L \times \hat{SL}(2,\mathbb{R})_R$$

↳ obtained in the WZW formulation by $\delta g = \omega(z)\, g + g\, \bar{\omega}(\bar{z})$

↳ or from the isometries of AdS_3 with worldsheet dependent parameters (e.g. $\delta u(z,\bar{z}) = \alpha(z)$)

These symmetries are generated by the chiral WZW currents

$$J^a(z) = -k\, Tr(T^a \partial g g^{-1}), \qquad \bar{J}^a(\bar{z}) = k\, Tr(\bar{T}^a g^{-1} \bar{\partial} g)$$

which are related to the Noether currents of NLSM by

$$j_{Noether}(z,\bar{z}) = J^a(z) + \theta(z,\bar{z}) \longrightarrow \text{"topological" term,}$$

Example: for translations along X^M ($\delta X^M = \xi^M$) we have

$$j_\xi = \ell_s^{-2}\, \xi^\nu \left(M_{\nu\mu} \bar{\partial} X^\mu \partial + M_{\mu\nu} \partial X^\mu \bar{\partial} \right)$$

$$\xi = \partial_u \quad \Rightarrow \quad j_{\partial u} = \underbrace{\frac{k}{2}(\partial u - 2r\partial v)\bar{\partial}}_{J^3} + \frac{k}{4}(\bar{\partial} u \partial - \partial u \bar{\partial})$$

For the $M=0$ BTZ black hole all Noether currents are chiral, ie.

$$j_{Noether} = \bar{J}^a \bar{\partial} \quad \text{or} \quad j_{Noether} = \bar{J}^a \partial$$

From the chiral part of $j_{Noether}$ or the WZW current we obtain:

$$J_n^a = \frac{1}{2\pi} \oint e^{inz} J^a \quad \Rightarrow$$

& particularly interested in J_0^3

$$[J_n^3, J_m^3] = -\frac{k}{2} n \delta_{n+m}$$

$$[J_n^3, J_m^\pm] = \pm J_{n+m}^\pm$$

$$[J_n^+, J_m^-] = -2 J_{n+m}^3 + k n \delta_{n+m}$$

Energy and angular momentum of the string ($\epsilon = \epsilon_L + \epsilon_R$, $p = \epsilon_L - \epsilon_R$)

$$\epsilon_L = \frac{1}{2\pi} \oint j \partial u = \underbrace{\frac{1}{2\pi} \oint J^3}_{J_0^3} + \underbrace{\frac{k}{2\pi} \oint \partial \sigma u}_{\text{winding}}, \qquad \epsilon_R = \frac{1}{2\pi} \oint j \partial v$$

The worldsheet stress energy tensor (at large k):

$$T_{ab} = \frac{2}{\sqrt{-\eta}} \frac{\delta S}{\delta \eta^{ab}} \quad \Rightarrow \quad T = -l_s^{-2} G_{\mu\nu} \partial X^\mu \partial X^\nu$$

Exercise. compute the WZW currents and verify that the stress tensor matches the Sugawara stress tensor at large k

$$T = \frac{1}{k-2} J^a J^a = \frac{1}{k-2} \left[\frac{1}{2} (J^+ J^- + J^- J^+) - (J^3)^2 \right]$$

Virasoro algebra:

$$L_n = -\frac{1}{2\pi} \oint T e^{inz}, \qquad [L_n, L_m] = (n-m) L_{n+m} + \frac{\tilde{c}}{12} n(n^2-1) \delta_{n+m},$$

$$\tilde{c} = \frac{3k}{k+2}$$

The spectrum

Spectrum of strings on AdS_3 → unitary reps of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

$$|j,m\rangle \longrightarrow C_2|j,m\rangle = -j(j-1)|j,m\rangle, \qquad J_0^3|j,m\rangle = m|j,m\rangle$$

$$C_2 \equiv \sum_a J_0^a \bar{J}_0^a, \qquad J_0^a \equiv \frac{1}{2\pi}\oint J^a,$$

(1) Principal discrete reps (short strings)

$$D_j^{\pm} = \{|j,m\rangle : m = \pm j, \pm(j+1), \pm(j+2), \dots\}, \qquad J_0^{\mp}|j,\pm j\rangle = 0$$

unitarity: $j \in \mathbb{R}_{>0}$

$$\frac{1}{2} < j < \frac{k}{2}$$

wavefunction is L^2 ← $\frac{1}{2} < j$; $j < \frac{k}{2}$ → no ghost theorem

(2) Principal continous reps (long strings)

$$C_j^{\lambda} = \{|j,m\rangle : m = \lambda, \lambda\pm1, \lambda\pm2, \dots\}, \qquad \lambda \in [0,1)$$

unitarity: $j = \frac{1}{2} + is, \quad s \in \mathbb{R}$

Virasoro constraints:

$$(L_0 - 1)|\psi\rangle = 0 \quad \Rightarrow \quad \underbrace{-\frac{j(j-1)}{k-2}}_{\frac{C_2}{k-2}} + N + h = 1$$

N: level; h: internal manifold

↳ for $D_j^{\pm}$: N is bounded from above (finite string excitations)

↳ for C_j^{λ}: only solution has $N = h = 0$ (the tachyon)

This can be remedied by the inclusion of additional "flowed" representations.

Spectral flow

Let's consider the EOM of the NLSM with general T_u, T_v:

$$\delta S = -\frac{1}{\ell_s^2}\int d^2z \Big\{ (f(r) + \bar\partial u\,\partial v)\,\delta r - \big[\tfrac{1}{2}\partial\bar\partial u + \bar\partial(r\partial v)\big]\delta u - \big[\tfrac{1}{2}\partial\bar\partial v + \partial(r\bar\partial u)\big]\delta v \Big\}$$

The EOM are invariant under the spectral flow transformation:

$$u \to u - w z\,, \qquad v \to v - w\bar z$$

In general (non AdS_3) spacetimes this is related to the existence of chiral currents.

Exercise: assume $M_{\mu\nu}$ is invariant under translations along X^m. Find the conditions on $M_{\mu\nu}$ such that $X^m \to X^m + w z$ leaves the EOM invariant.

We can use spectral flow to generate winding string solutions:

$$u(\tau),\ v(\tau),\ r(\tau) \xrightarrow{\ SF\ } \tilde u(\tau,\sigma),\ \tilde v(\tau,\sigma)$$

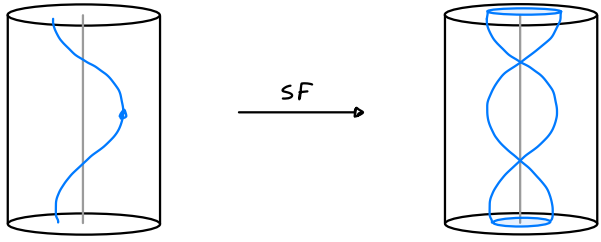
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\text{EOM} \Rightarrow \text{geodesic eq.} \qquad\qquad \tilde\psi = \frac{\tilde u - \tilde v}{2} = \frac{u-v}{2} - w\sigma$$

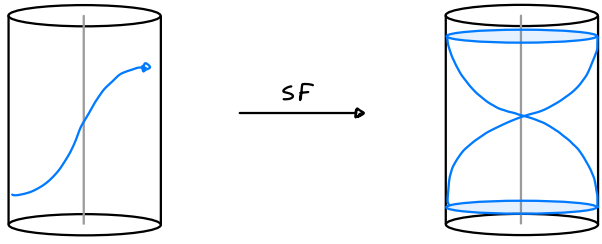
$$\Downarrow$$

$$\tilde\psi \sim \tilde\psi - 2\pi w$$

short strings:



long strings:



The spectral flow transformation induces a transformation of the currents that leaves the algebra unchanged, i.e. it's an automorphism of the $SL(2,\mathbb{R})$ algebra:

easy to check using the results of a previous exercise.

$$J^3 \to \tilde{J}^3 = J^3 - \frac{k}{2}\omega, \qquad J^\pm \to \tilde{J}^\pm = e^{\mp i 2\omega} J^\pm, \qquad \omega \in \mathbb{Z}$$

When the group is non compact, spectral flow generates new reps.

Hence, for $SL(2,\mathbb{R})$ we have:

improved no-ghost theorem

$$D_j^{\pm,\omega}, \quad C^{\lambda,\omega}_{\frac{1}{2}+is} \quad \text{where} \quad \frac{1}{2} < j < \frac{k-1}{2}$$

Flowed stress tensor:

$$T = \frac{1}{k}\left[\frac{1}{2}(J^+J^- + J^-J^+) - (J^3)^2\right] \quad \Rightarrow \quad \tilde{T} = T + \omega J^3 - \frac{k\omega^2}{4}$$

$$\Rightarrow \quad \tilde{L}_0 = \frac{1}{2\pi}\oint \tilde{T} = L_0 + \omega \bar{J}_0^3 - \frac{k\omega^2}{4}$$

The Virasoro constraints of a flowed state $|j,m\rangle \longrightarrow |\tilde{j},\tilde{m}\rangle$ become

$$(L_0 - 1)|\tilde{\psi}\rangle = 0 \quad \Rightarrow \quad -\frac{\tilde{j}(\tilde{j}-1)}{k} - w\tilde{J}_0^3 - \frac{kw^2}{4} + N + h = 1$$

- arbitrarily large
- w-flowed continuous reps.

Recall the left moving energy of the string is given by

$$\epsilon_L^{(w)} = \frac{1}{2\pi}\oint j_{2u} = \tilde{J}_0^3 + \frac{k}{4}w$$

The Virasoro constraints become

$$-\frac{\tilde{j}(\tilde{j}-1)}{k} - w\,\epsilon_L^{(w)} = 1 \quad \Rightarrow \quad \epsilon_L^{(1)} = -\frac{\tilde{j}(\tilde{j}-1)}{k} - 1$$

$$\Rightarrow \quad \underbrace{\epsilon_L^{(w)} = \frac{1}{w}\,\epsilon_L^{(1)}}$$

energy of a twist-w state in a symmetric orbifold !

Comments :

- For $D_{\tilde{j}}^{\pm}$, $\epsilon_L^{(1)} = -\frac{\tilde{j}(\tilde{j}-1)}{k} - 1$ can not be generically solved since

$$\tilde{J}_0^3\,|\tilde{j},\tilde{m}\rangle = \tilde{m}\,|\tilde{j},\tilde{m}\rangle, \quad \tilde{m} \in \mathbb{Z}$$

- For $C_{\tilde{j}}^{\hat{}}$, we have $\tilde{m} = \lambda + n$ with $\lambda \in \mathbb{R}$, $n \in \mathbb{Z}$. Hence, it's possible to solve the Virasoro constraints !

  => spectrum of long strings captured by a symmetric orbifold where w = twist of the state ($w=1 \rightarrow$ untwisted sector)

However, the holographic dual is not a symmetric orbifold

in tension with the semiclassical limit ($k \gg 1$)

of sugra: no tower of higher spins fields

At $k=1$ we don't have a semiclassical approximation but:

(1) there's an ∞ tower of massless higher spin states

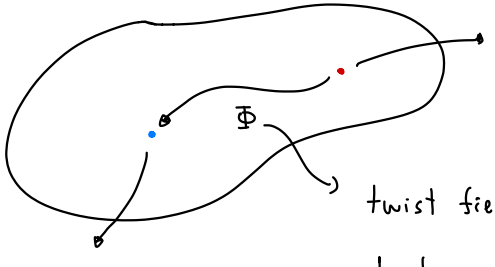
(2) there are no discrete reps

(3) only the $s=0$ $C\hat{}_j$ reps are present

discrete spectrum

$\Rightarrow$ At $k=1$ the holographic dual is $\mathrm{Sym}^p T^4$ with $c = 6p$

For general k we expect to move along the conformal manifold



CFT with $c = 6kp$

twist field (with $n=2$) that partially destroys the symmetric product structure

↳ long string spectrum still captured by a symmetric orbifold

↳ allows for the definition of single-trace deformations