Lecture 2

In the previous lecture we motivated the the $T\bar{T}$ deformation from both quantum field theory and quantum gravity points of view. In particular, we showed that the $T\bar{T}$ deformation is an irrelevant deformation driven by a specific combination of the (product of) components of the stress tensor. In this lecture we will show that this combination guarantees that the expectation value of the $T\bar{T}$ operator is finite and that it has a fixed scaling dimension. We will then show that these properties, which underlie the solvability of the $T\bar{T}$ deformation, can be generalized to an infinite class of irrelevant deformations.

2 The $T\bar{T}$ operator

We will now establish two crucial properties of the $T\bar{T}$ operator, namely

(*a*) the fact that this operator is well defined, i.e. that the coincident-point limit below is finite (at least within expectation values)

$$T\bar{T} = \lim_{y^\pm \to x^\pm} \big[T_{++}(x^\pm)T_{--}(y^\pm) - T_{+-}(x^\pm)T_{+-}(y^\pm)\big]; \tag{2.1}$$

(*b*) the expectation value of the $T\bar{T}$ operator is constant and factorizes, meaning that

$$\langle T\bar{T}\rangle = \langle T_{++}(x^\pm)\rangle\langle T_{--}(x^\pm)\rangle - \langle T_{+-}(x^\pm)\rangle^2 = \text{constant}. \tag{2.2}$$

This implies, in particular, that the scaling dimension of the $T\bar{T}$ operator is always 4.

In order to prove these statements, we need to make several assumptions on the local and global properties of the QFTs we're interested in. For convenience we work in Euclidean signature obtained by letting $t \to it$ such that

$$x^+ = x + t \to x + it =: z, \qquad x^- = x - t \to x - it =: \bar{z}. \tag{2.3}$$

Likewise, we introduce the following notation for the components of the stress tensor

$$T := -2\pi T_{zz}, \qquad \bar{T} := -2\pi T_{\bar{z}\bar{z}}, \qquad \theta := 2\pi T_{z\bar{z}}. \tag{2.4}$$

The assumptions we will make are

(*i*) Local translational and rotational symmetry: as described before, we assume the QFTs we're interested in are invariant under Poincaré transformations, consequence of which

$$\partial_\mu T^\mu{}_\nu = 0, \qquad T_{\mu\nu} = T_{\nu\mu}. \tag{2.5}$$

We can write the conservation law explicitly as follows

$$\partial_\mu T^\mu{}_\nu = 0 \qquad \Longrightarrow \qquad \begin{aligned} \partial_z T_{\bar{z}z} + \partial_{\bar{z}} T_{zz} = 0, \\ \partial_{\bar{z}} T_{z\bar{z}} + \partial_z T_{\bar{z}\bar{z}} = 0, \end{aligned} \tag{2.6}$$

where we used the fact that in the $(z, \bar{z})$ coordinates the line element is $ds^2 = dzd\bar{z}$ such that the components of the metric are $g_{zz} = g_{\bar{z}\bar{z}} = g^{zz} = g^{\bar{z}\bar{z}} = 0$, $g_{z\bar{z}} = 1/2$, and $g^{z\bar{z}} = 2$.

(*ii*) Local interactions: we assume that all interactions are local in the sense that, at long enough distances, the connected part of any two-point function vanishes and we're left only with the disconnected part

$$\lim_{z' \to \infty} \langle \mathcal{O}(z)\mathcal{O}'(z')\rangle = \langle \mathcal{O}(z)\rangle\langle \mathcal{O}'(z')\rangle, \qquad \forall \quad \mathcal{O}, \mathcal{O}'. \tag{2.7}$$

(*iii*) Global translational symmetry: we assume that the translational symmetry is not spontaneously broken in the sense that, up to a phase factor,

$$U(z_0)|0\rangle = e^{iTz_0} e^{i\bar{T}\bar{z}_0}|0\rangle \approx |0\rangle, \tag{2.8}$$

where $U(z_0)$ generates the translation $(z, \bar{z}) \to (z + z_0, \bar{z} + \bar{z}_0)$ and $\approx$ denotes equality up to a phase factor. This assumption implies that the one and two-point functions of any field $\mathcal{O}(z)$ satisfy

$$\begin{aligned} \langle \mathcal{O}(z)\rangle &= \langle U^\dagger(z - z_0)\mathcal{O}(z_0)U(z - z_0)\rangle = \langle \mathcal{O}(z_0)\rangle, \\ \langle \mathcal{O}(z)\mathcal{O}(z')\rangle &= \langle U^\dagger(z - z_0)\mathcal{O}(z_0)\mathcal{O}(z' - z + z_0)U(z - z_0)\rangle = \langle \mathcal{O}(z_0)\mathcal{O}(z' - z + z_0)\rangle, \end{aligned}$$

which in turn imply that

$$\langle \mathcal{O}(z)\rangle = \text{constant}, \qquad \langle \mathcal{O}(z)\mathcal{O}(z')\rangle = f(z - z'). \tag{2.9}$$

The assumptions (*i*) – (*iii*) imply that the underlying geometry is either the plane or the cylinder. We don't have a general definition of the $T\bar{T}$ operator in other spacetimes.

Exercise 2.1: consider the case when the QFT is conformally invariant, i.e. a CFT. Show that properties (*ii*) and (*iii*) hold for $\mathcal{O} = T$ both on the plane and the cylinder. Hint: exploit the fact that theory is conformal to map the plane to the cylinder.

2.1 Finiteness and factorizability of $T\bar{T}$

Let us now show that the $T\bar{T}$ operator, formally defined by[2]

$$T\bar{T}(z') := \lim_{z \to z'} \left[T(z)\bar{T}(z') - \theta(z)\theta(z')\right], \tag{2.10}$$

[2]For simplicity, we henceforth drop the dependence of $T(z, \bar{z})$, $\bar{T}(z, \bar{z})$, and $\theta(z, \bar{z})$ on the $\bar{z}$ coordinate.

is well defined (free of divergences) in any QFT satisfying assumptions (i) – (iii), up to total derivative terms that don't affect its expectation value. In order to see this let's consider the operator product expansion (OPE) for the different components of the stress tensor. Recall that the OPE tells us about the short distance behavior between two operators,

$$\mathcal{O}(z)\mathcal{O}(z') \approx \sum_i c_i(z-z')\mathcal{O}_i(z'), \tag{2.11}$$

where the sum runs over all of the local operators $\mathcal{O}_i(z)$ of the theory, including operators obtained from derivatives, e.g. $\partial_z \mathcal{O}_i(z)$. In the OPE (2.11), $\approx$ indicates that the RHS is given up to analytic terms of the form $(z-z')^n$ with $n > 1$ such that the $c_i(z-z')$ depend on $(z-z')^n$ with $n \leq 0$. Furthermore, note that since the theory is invariant under translations, the RHS can only depend on the difference $z - z'$.

The components of the stress tensor in any Poincaré-invariant QFT satisfy the following generic OPEs

$$\begin{pmatrix} T(z)\bar{T}(z') & T(z)\theta(z') \\ \theta(z)\bar{T}(z') & \theta(z)\theta(z') \end{pmatrix} \approx \sum_i \begin{pmatrix} D_i(z-z') & A_i(z-z') \\ B_i(z-z') & C_i(z-z') \end{pmatrix} \mathcal{O}_i(z'). \tag{2.12}$$

As a result, the OPE between the terms featured in the definition of the $T\bar{T}$ operator reads

$$I(z,z') \coloneqq T(z)\bar{T}(z') - \theta(z)\theta(z') \approx \sum_i \big[D_i(z-z') - C_i(z-z')\big]\mathcal{O}_i(z'). \tag{2.13}$$

Since we are summing over all local operators on the RHS, we can split the sum as follows

$$I(z,z') \approx \mathcal{O}_{T\bar{T}}(z') + \sum_i G_i(z-z')\mathcal{O}_i(z') + \sum_i F_i^{\mu}(z-z')\partial_\mu \mathcal{O}_i(z'), \tag{2.14}$$

where the first term is the only operator for which $D_i(z-z') - C_i(z-z')$ is constant, a constant that has been absorbed into the definition of $\mathcal{O}_{T\bar{T}}(z)$. The second sum in (2.14) runs over operators without derivatives while the third sum includes terms with at least one derivative. Both the $G_i(z-z')$ and $F_i^{\mu}(z-z')$ terms are potentially divergent in the limit $z \to z'$. Our goal will be to show that $G_i(z-z') = 0$ as a consequence of assumptions (i) – (iii). On the other hand, while we cannot show that $F_i^{\mu}(z-z')$ vanishes, these terms do not contribute to the expectation value of the $T\bar{T}$ operator since, by assumption (iii), we have

$$\langle \partial_\mu \mathcal{O}_i(z)\rangle = \partial_\mu \langle \mathcal{O}_i(z)\rangle = 0. \tag{2.15}$$

We begin by considering derivatives of the operator $I(z,z')$ defined in (2.13). The conservation of the stress tensor implies that

$$\partial_{\bar{z}} I(z,z') = \underbrace{\partial_{\bar{z}} T(z)}_{=\partial_z \theta(z)} \bar{T}(z') - \partial_{\bar{z}}\theta(z)\theta(z') + \theta(z)\big[\underbrace{\partial_{z'}\bar{T}(z') - \partial_{\bar{z}'}\theta(z')}_{=0}\big]$$

$$= (\partial_z + \partial_{z'})\theta(z)\bar{T}(z') - (\partial_{\bar{z}} + \partial_{\bar{z}'})\theta(z)\theta(z'), \tag{2.16}$$

where the last term in the first line has been added by hand since it vanishes exactly. Similarly, it's not difficult to show that

$$\partial_z I(z, z') = (\partial_z + \partial_{z'})T(z)\bar{T}(z') - (\partial_{\bar{z}} + \partial_{\bar{z}'})T(z)\theta(z'). \tag{2.17}$$

Both (2.16) and (2.17) feature $(\partial_z + \partial_{z'})$ and $(\partial_{\bar{z}} + \partial_{\bar{z}'})$ derivatives which satisfy

$$(\partial_z + \partial_{z'})f(z - z') = 0, \qquad (\partial_{\bar{z}} + \partial_{\bar{z}'})\bar{f}(\bar{z} - \bar{z}') = 0, \tag{2.18}$$

for any functions $f(z - z')$ and $\bar{f}(\bar{z} - \bar{z}')$. Consequently, the conservation of the stress tensor implies that

$$\partial_z\big[T(z)\bar{T}(z') - \theta(z)\theta(z')\big] = \sum_i \big[B_i(z - z')\partial_{z'}\mathcal{O}_i(z') - C_i(z - z')\partial_{\bar{z}'}\mathcal{O}_i(z')\big], \tag{2.19}$$

$$\partial_{\bar{z}}\big[T(z)\bar{T}(z') - \theta(z)\theta(z')\big] = \sum_i \big[D_i(z - z')\partial_{z'}\mathcal{O}_i(z') - A_i(z - z')\partial_{\bar{z}'}\mathcal{O}_i(z')\big]. \tag{2.20}$$

Integrating these expressions we learn that $G_i(z - z') = 0$, so we have

$$T(z)\bar{T}(z') - \theta(z)\theta(z') \approx \mathcal{O}_{T\bar{T}}(z') + \sum_i F_i^{\mu}(z - z')\partial_\mu\mathcal{O}_i(z'). \tag{2.21}$$

We thus find that the expectation value of the $T\bar{T}$ operator is finite in any QFT satisfying assumptions *(i)* – *(iii)* such that

$$\langle T\bar{T}(z')\rangle = \lim_{z\to z'}\langle T(z)\bar{T}(z') - \theta(z)\theta(z')\rangle = \langle\mathcal{O}_{T\bar{T}}(z')\rangle. \tag{2.22}$$

Factorizability of $T\bar{T}$

Let us now show that $\langle I(z, z')\rangle = \langle T(z)\bar{T}(z') - \theta(z)\theta(z')\rangle$ factorizes in any QFT satisfying assumptions *(i)* – *(iii)*. First, we show that $\langle I(z, z')\rangle$ is constant,

$$\begin{aligned}
\partial_{\bar{z}}\langle I(z, z')\rangle &= \langle\underbrace{\partial_{\bar{z}}T(z)}_{\partial_z\theta(z)}\bar{T}(z')\rangle - \partial_{\bar{z}}\langle\theta(z)\theta(z')\rangle \\
&= \partial_z\langle\theta(z)\bar{T}(z')\rangle - \partial_{\bar{z}}\langle\theta(z)\theta(z')\rangle \\
&= -\partial_{z'}\langle\theta(z)\bar{T}(z')\rangle + \partial_{\bar{z}'}\langle\theta(z)\theta(z')\rangle \\
&= -\langle\theta(z)\big[\underbrace{\partial_{z'}\bar{T}(z') - \partial_{\bar{z}'}\theta(z')}_{=0}\big]\rangle \\
&= 0,
\end{aligned} \tag{2.23}$$

where we used the conservation of the stress tensor in the first and fourth lines, and used assumption *(iii)* on the third line to exchange $(\partial_z, \partial_{\bar{z}})$ with $(-\partial_{z'}, -\partial_{\bar{z}'})$. Similarly, it's not difficult to show that $\partial_z \langle I(z,z')\rangle = 0$. Then, using assumption *(iii)* once again, we find that $\langle I(z,z')\rangle$ is also annihilated by $\partial_{z'}$ and $\partial_{\bar{z}'}$ such that

$$\langle I(z,z')\rangle = \text{constant}. \tag{2.24}$$

Since $\langle I(z,z')\rangle$ is constant, we can put z and z' at any location. In particular, we can choose z and z' to be infinitely separated so that, by assumption *(ii)*, we have

$$\langle I(z,z')\rangle = \langle T(z)\bar{T}(z')\rangle - \langle \theta(z)\theta(z')\rangle\Big|_{|z-z'|\to\infty} \tag{2.25}$$

$$= \langle T(z)\rangle\langle \bar{T}(z')\rangle - \langle \theta(z)\rangle\langle \theta(z')\rangle. \tag{2.26}$$

We conclude that the $T\bar{T}$ operator is a well-defined operator that factorizes in any Poincaré-invariant QFT in two dimensions. This implies, in particular, that the $T\bar{T}$ operator has fixed scaling dimension 4.

$T\bar{T}$ in CFT

In order to gain some intuition, let us now consider a QFT that, in addition to being Poincaré invariant, is also invariant under scale transformations. In two dimensions, scale invariance implies conformal invariance, so we are considering a full-fledged CFT. In this case, it's possible to write down an explicit expression for the $T\bar{T}$ operator and, using the state-operator correspondence, identify it with a state in the Hilbert space of the theory.

A defining characteristic of a CFT is that the stress tensor is traceless, such that

$$T_{z\bar{z}} = 0 \qquad \Longrightarrow \qquad \partial_{\bar{z}} T(z) = 0, \qquad \partial_z \bar{T}(\bar{z}) = 0. \tag{2.27}$$

The chiral conservation of the stress tensor implies the existence of an infinite number of local symmetries described by two commuting copies of the Virasoro algebra (see exercise 1.2). On the plane, the generators of this algebra are related to the stress tensor by

$$T(z) = \sum_n z^{-n-2} L_n, \qquad \bar{T}(\bar{z}) = \sum_n \bar{z}^{-n-2} \bar{L}_n, \tag{2.28}$$

and can be shown to satisfy

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m}, \tag{2.29}$$

$$[\bar{L}_n, \bar{L}_m] = (n-m)\bar{L}_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m}, \tag{2.30}$$

$$[L_n, \bar{L}_m] = 0, \tag{2.31}$$

where c is the central charge of the CFT.

The fact that L_n and $\bar{L}_m$ generators commute with each other can be understood as a consequence of the vanishing of the $T(z)\bar{T}(\bar{z})$ OPE. Another consequence of the vanishing of the $T(z)\bar{T}(\bar{z})$ OPE is that the $T\bar{T}$ operator is manifestly finite such that

$$T\bar{T}(z) = \lim_{z'\to z} T(z')\bar{T}(\bar{z}) = \sum_{n,m} z^{-n-2}\bar{z}^{-m-2}L_{-n}\bar{L}_{-m}. \tag{2.32}$$

This implies, in particular, that the expectation value of the $T\bar{T}$ operator automatically factorizes in a CFT. We furthermore note that the components of the stress tensor $T(z)$ and $\bar{T}(\bar{z})$ are related via the state-operator correspondence to descendant states of the vacuum, namely

$$T(z) \;\leftrightarrow\; L_{-2}|0\rangle, \qquad \bar{T}(\bar{z}) \;\leftrightarrow\; \bar{L}_{-2}|0\rangle. \tag{2.33}$$

The $T\bar{T}$ operator is also associated with a descendant of the vacuum. Indeed, using (2.32) and (2.31) we identify the $T\bar{T}$ operator with

$$T\bar{T}(z) \;\leftrightarrow\; L_{-2}\bar{L}_{-2}|0\rangle. \tag{2.34}$$

Exercise 2.2: a CFT contains an infinite number of descendant states obtained by acting the L_{-n} and $\bar{L}_{-m}$ operators on the vacuum with $n, m \geq 2$. In particular, every CFT contains the state $L_{-3}\bar{L}_{-3}|0\rangle$. What's the interpretation of this state in terms of the stress tensor? Can this operator be generalized to QFTs without scale invariance? Hint: use the Virasoro algebra (2.29) and the mode expansion of $T(z)$, $\bar{T}(z)$ in (2.28).

Generalization to other states

We have shown that the vacuum expectation value of the $T\bar{T}$ operator is finite and factorizes. These properties of the $T\bar{T}$ operator continue to hold for more general states. Indeed, for non-degenerate eigenstates $|n\rangle$ of the energy and momentum we have

$$\begin{aligned}\langle n|T\bar{T}|n\rangle &= \lim_{z\to z'}\left[\langle n|T(z)\bar{T}(z')|n\rangle - \langle n|\theta(z)\theta(z')|n\rangle\right]\\ &= \langle n|T|n\rangle\langle n|\bar{T}|n\rangle - \langle n|\theta|n\rangle^2 = \text{constant}.\end{aligned} \tag{2.35}$$

In order to show this we need to revisit the assumptions *(i)* – *(iii)* described earlier. Assumption *(i)* (local translation invariance) still holds by definition of the theories under consideration. Assumption *(iii)* (global translation invariance) also holds. This follows from the fact that $|n\rangle$ is an eigenstate of the energy and momentum such that

$$U(z_0)|n\rangle = e^{iE_L z_0}e^{iE_R\bar{z}_0}|n\rangle, \tag{2.36}$$

where the left and right-moving energies E_L and E_R are defined by

$$E_L := E + P, \qquad E_R := E - P. \tag{2.37}$$

Consequently, we find that the one and two-point functions in the state $|n\rangle$ still satisfy

$$\langle n|\mathcal{O}(z)|n\rangle = \text{constant}, \qquad \langle n|\mathcal{O}(z)\mathcal{O}(z')|n\rangle = f(z-z'). \tag{2.38}$$

Let $\langle I_n(z,z')\rangle \coloneqq \langle n|T(z)\bar{T}(z')|n\rangle - \langle n|\theta(z)\theta(z')|n\rangle$. Then, repeating the same steps used in the case where $|n\rangle = |0\rangle$ together with assumptions *(i)* and *(iii)* we learn that

$$\langle I_n(z,z')\rangle = \text{constant}. \tag{2.39}$$

We now note that we cannot use assumption *(ii)* (local interactions) to argue that $\langle I_n(z,z')\rangle$ factorizes. This is because $\langle n|\mathcal{O}(z)\mathcal{O}(z')|n\rangle$ can be understood as a four-point function with two operators inserted at $z=0$ and $z=\infty$. In other words, the states $|n\rangle$ and $\langle n|$ are not localized, so there is no sense in which taking $|z-z'|\to\infty$ turns off the interactions between $\mathcal{O}(z)$ and $\mathcal{O}(z')$. Nevertheless, since $|n\rangle$ is an energy and momentum eigenstate, we can use assumption *(iii)* to write

$$\begin{aligned}\langle I_n(z,z')\rangle &= \sum_{n'}\left(\langle n|T(z)|n'\rangle\langle n'|\bar{T}(z')|n\rangle - \langle n|\theta(z)|n'\rangle\langle n'|\theta(z')|n\rangle\right)\\ &= \sum_{n'}\Big(\langle n|T(z)|n'\rangle\langle n'|\bar{T}(z)|n\rangle e^{i[E_L(n)-E_L(n')](z'-z)}e^{i[E_R(n)-E_R(n')](z'-z)}\\ &\qquad\qquad - \langle n|\theta(z)|n'\rangle\langle n'|\theta(z)|n\rangle e^{i[E_L(n)-E_L(n')](z'-z)}e^{i[E_R(n)-E_R(n')](z'-z)}\Big).\end{aligned} \tag{2.40}$$

The phases in (2.40) are a result of using the translation operator to move the operators $\mathcal{O}(z) = \{\bar{T}(z),\theta(z)\}$ from $(z',\bar{z}')$ to $(z,\bar{z})$, that is

$$\begin{aligned}\langle n'|\mathcal{O}(z')|n\rangle &= \langle n'|U^\dagger(z'-z)\mathcal{O}(z)U(z'-z)|n\rangle\\ &= \langle n'|\mathcal{O}(z)|n\rangle e^{i[E_L(n)-E_L(n')](z'-z)}e^{i[E_R(n)-E_R(n')](z'-z)}.\end{aligned} \tag{2.41}$$

The sum in (2.40) can be split into the contribution of the state with $n'=n$ and the remainder, such that

$$\langle I_n(z,z')\rangle = \underbrace{\langle n|T(z)|n\rangle\langle n|\bar{T}(z)|n\rangle - \langle n|\theta(z)|n\rangle\langle n|\theta(z)|n\rangle}_{\text{constant}} + \sum_{n'\neq n} F_n(z-z'), \tag{2.42}$$

for some function $F_n(z-z')$. The first term is a constant as a consequence of (2.38) while the second term depends on the coordinates. Since $\langle I_n(z,z')\rangle$ is a constant and $|n\rangle$ is non-degenerate, all of the terms in $\sum_{n'\neq n}F_n(z-z')$ must cancel each other. As a result, we find that the expectation value of the $T\bar{T}$ operator in a non-degenerate state $|n\rangle$ also factorizes

$$\langle n|T\bar{T}|n\rangle = \langle n|T|n\rangle\langle n|\bar{T}|n\rangle - \langle n|\theta|n\rangle^2 = \text{constant}. \tag{2.43}$$

We conclude that the expectation value of the $T\bar{T}$ operator is also finite and factorizes when the vacuum is replaced by an energy and momentum eigenstate.

2.2 Higher spin generalizations

The feature that makes the expectation value of the $T\bar{T}$ operator finite and factorizable is that it's built from a particular combination of the components of the stress tensor. The properties that make the $T\bar{T}$ operator turn out to be more universal and can be generalized to conserved currents other than the stress tensor as we will now describe.

Let us consider a theory with additional conserved currents. We introduce the following notation

$$\mathcal{O}_{h,\bar{h}}: \qquad \text{scaling dimension:} \quad h+\bar{h}, \qquad \text{spin:} \quad h-\bar{h}. \tag{2.44}$$

We are interested in conserved currents of scaling dimension $s+1$. Let us single out the following components of these currents

component	spin
$T_{s+1} \coloneqq T_{s+1,0}$	$s+1$
$\theta_{s-1} \coloneqq \theta_{s,1}$	$s-1$
$\bar{T}_{s+1} \coloneqq \bar{T}_{0,s+1}$	$-(s+1)$
$\bar{\theta}_{s-1} \coloneqq \bar{\theta}_{1,s}$	$-(s-1)$

(2.45)

which are assumed to satisfy the following conservation laws

$$\partial_{\bar{z}} T_{s+1}(z) = \partial_z \theta_{s-1}(z), \qquad \partial_z \bar{T}_{s+1}(z) = \partial_{\bar{z}} \bar{\theta}_{s-1}(z). \tag{2.46}$$

In differential notation we can alternatively write

$$j_{s+1} \coloneqq T_{s+1} dz - \theta_{s-1} d\bar{z}, \qquad \star j_{s+1} = T_{s+1} dz + \theta_{s-1} d\bar{z}, \qquad d \star j_{s+1} = 0, \tag{2.47}$$

$$\bar{j}_{s+1} \coloneqq \bar{\theta}_{s-1} dz - \bar{T}_{s+1} d\bar{z}, \qquad \star \bar{j}_{s+1} = \bar{\theta}_{s-1} dz + \bar{T}_{s+1} d\bar{z}, \qquad d \star \bar{j}_{s+1} = 0. \tag{2.48}$$

The conservation of these currents leads to conserved charges

$$P_s = \int_c \star j_{s+1}, \qquad \bar{P}_s = \int_c \star \bar{j}_{s+1}, \tag{2.49}$$

which are invariant under deformations of the contour c.

Exercise 2.3: verify that the conservation law (2.46) is compatible with the scaling dimension and spin assignments made in (2.45).

For example, for a $U(1)$ current, the scaling dimension is 1 ($s=0$) and we can write

$$J_z = T_1, \qquad J_{\bar{z}} = -\bar{T}_1 \qquad \Longrightarrow \qquad \partial_\mu J^\mu = 0. \tag{2.50}$$

In this case, two of the components $(\theta_{-1}, \bar{\theta}_{-1})$ in (2.45) turn out to be redundant and can be set to zero or assigned to some other $U(1)$ current. On the other hand, when $s > 0$, we need at least four components of the current to satisfy the conservation laws, as we have already observed for the case of the stress tensor where $s = 1$.

For each of the higher spin currents described above we can define the following scalar

$$X_s \coloneqq \lim_{z \to z'} \left[T_{s+1}(z)\bar{T}_{s+1}(z') - \theta_{s-1}(z)\bar{\theta}_{s-1}(z') \right]. \tag{2.51}$$

In terms of the currents j_{s+1} and $\bar{j}_{s+1}$, this can be written more compactly as

$$X_s\, dz \wedge d\bar{z} = \lim_{z \to z'} j_{s+1}(z) \wedge \bar{j}_{s+1}(z'). \tag{2.52}$$

We see that the $T\bar{T}$ operator $X_1 = T\bar{T}$ is one of an infinite family of operators with scaling dimension $2(s+1)$. When $s = 0$, the operator X_0 is marginal and made of the product of two $U(1)$ currents. On the other hand, when $s \geq 1$, all of the X_s operators are irrelevant. Hence, deforming a QFT by any of the X_s operators with $s \geq 1$ changes the UV behavior of the theory.

Following the same steps described in the previous section, it's not difficult to show that the X_s operators are finite and factorize such that

$$\langle n|X_s|n\rangle = \langle n|T_{s+1}(z)|n\rangle\langle n|\bar{T}_{s+1}(z)|n\rangle - \langle n|\theta_{s-1}(z)|n\rangle\langle n|\bar{\theta}_{s-1}(z)|n\rangle = \text{constant}. \tag{2.53}$$

This implies that the X_s operators have fixed scaling dimension $2(s+1)$, which allows us to define an instantaneous deformation analogous to the $T\bar{T}$ deformation (1.27).

Thus far we have assumed that the theory is Lorentz invariant (or rotationally invariant in Euclidean signature). In particular, we have required the X_s operators to be scalars, which is necessary to preserve the Lorentz symmetry under their deformation. However, it turns out that Lorentz invariance is not strictly necessary although it does simplify the analysis. The two simplest ways in which we can give up Lorentz invariance are:

(*i*) breaking Lorentz invariance of the undeformed QFT, case in which $T_{\mu\nu} \neq T_{\nu\mu}$. In this case $\theta_{s-1} \neq \bar{\theta}_{s-1}$, so the more general results presented in this section are immediately applicable to this case. Relaxing the $T_{\mu\nu} = T_{\nu\mu}$ property of the stress tensor is necessary, for example, to describe the $T\bar{T}$ deformation of non-relativistic QFTs.

(*ii*) breaking Lorentz invariance of the deformed QFT by letting the deforming operator $\tilde{X}_s$ transform under Lorentz transformations, i.e. by relaxing the condition that $\tilde{X}_s$ is a scalar. The simplest and most well studied example of this approach is the so-called $J\bar{T}$ deformation which is driven by the $J\bar{T}$ operator

$$J\bar{T} \coloneqq \lim_{z \to z'} \left[J(z)\bar{T}(z') - \bar{J}(z)\theta(z') \right], \tag{2.54}$$

where $J(z)$ and $\bar{J}(z)$ are two components of a $U(1)$ current, i.e. $(J, \bar{J}) = (J_z, J_{\bar{z}})$ in the notation of (2.50). This deformation has scaling dimension 3 and is therefore the simplest irrelevant deformation we can construct. This deformation is related to the holographic description of warped AdS_3 spacetimes, an interesting class of spacetimes found in the near-horizon region of extremal Kerr black holes.

Note that the $J\bar{T}$ and X_s deformations with $s \neq 1$ are less universal than the $T\bar{T}$ deformation since they require the existence of additional conserved currents in the undeformed QFT. In other words, the $T\bar{T}$ operator is a universal operator present in any Poincaré-invariant two-dimensional QFT, whereas the $J\bar{T}$ and $X_{s\neq 1}$ operators depend on the details of the QFT.

Exercise 2.4: verify that the $J\bar{T}$ operator is well defined and factorizes in any translationally invariant QFT satisfying assumptions *(i)* – *(iii)*.

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