Lecture 5

In the previous lecture we studied the modular invariance of the torus partition function of two-dimensional CFTs. In particular, we showed that modular invariance leads to universal expressions for the partition function and the density of high energy states. In this lecture we will show that the $T\bar{T}$ partition function is also invariant under modular transformations provided that the deformation parameter transforms in an appropriate way. The modular invariance of the partition function will be used in the next lecture to derive a universal expression for the partition function and the asymptotic density of states of $T\bar{T}$-deformed CFTs.

3.2 Modular invariance of $T\bar{T}$-deformed CFTs

The torus partition function of $T\bar{T}$-deformed CFTs is naturally defined by[7]

$$Z(\tau,\bar{\tau};\hat{\mu}) = \mathrm{Tr}\big(e^{-\beta H + i\theta P}\big) = \sum_{E_n, P_n} c(E_n, P_n)\big(e^{-\beta E_n(\hat{\mu}) + i\theta P_n}\big), \tag{3.37}$$

where $\hat{\mu} := \mu/R^2$ is the dimensionless deformation parameter and $E_n(\hat{\mu})$ is the deformed energy given in (2.86). Although we have set $R = 1$ in this section, it is important to distinguish between the dimensionless and dimensionful deformation parameters for reasons that will become clear shortly. In analogy with the CFT partition function (3.3), we can write the $T\bar{T}$-deformed one in terms of $(\tau,\bar{\tau})$ and the left/right-moving energies defined in (2.37) as follows

$$Z(\tau,\bar{\tau};\hat{\mu}) = \mathrm{Tr}\big(q^{E_L(\hat{\mu})}\,\bar{q}^{E_R(\hat{\mu})}\big), \qquad q = e^{2\pi i\tau}. \tag{3.38}$$

Note that the $T\bar{T}$ deformation breaks the conformal symmetry of the CFT, i.e. a $T\bar{T}$-deformed CFT is not a CFT (despite what the name might suggest) so it's not useful to write (3.38) in terms of the conformal weights $(h_n, \bar{h}_n)$.

The partition function (3.38) turns out to be modular invariant provided that the dimensionless deformation parameter changes under modular transformations in the following way

$$Z(\gamma\tau,\gamma\bar{\tau};\gamma\hat{\mu}) = Z(\tau,\bar{\tau};\hat{\mu}), \qquad \gamma\tau = \frac{a\tau+b}{c\tau+d}, \qquad \gamma\hat{\mu} = \frac{\hat{\mu}}{|c\tau+d|^2}. \tag{3.39}$$

The modular invariance of the partition function has been derived, as we will discuss below, using the explicit expression for the spectrum of $T\bar{T}$-deformed CFTs. This is in contrast to the modular invariance of the undeformed partition function, which is a consequence of the scaling symmetry of the CFT. The modular invariance of the $T\bar{T}$ partition function is not obviously related to symmetries of the deformed theory and it would be interesting to try to understand it in this way. Nevertheless we can provide an intuitive understanding of why (3.39) is reasonable. We first note that it's the dimensionless deformation parameter $\hat{\mu}$, as opposed to the dimensionful one μ, that changes under modular transformations. This can be

[7] From now on, the deformation parameter μ will be assumed to be positive, and bounded from above by (2.93) in order to avoid the appearance of complex energy states and high and low energies.

understood as a consequence of the change of the size of the spatial circle since, under modular transformations, we have

$$R^2 \to \frac{R^2}{|c\tau + d|^2}. \tag{3.40}$$

The comment above does not explain why the partition function is modular invariant in the first place. One way to see this is to recall that the $T\bar{T}$ operator has a fixed scaling dimension in any Poincaré-invariant QFT. We can therefore promote the deformation parameter to a *spurion*, i.e. to a dynamical field, with scaling dimension -2, such that

$$\mu \to \varphi, \quad \text{with} \quad h = \bar{h} = -1. \tag{3.41}$$

In this way, the infinitesimal deformation of the action (1.28) becomes

$$\delta I = 8\pi\mu \int d^2x (T\bar{T})_0 \quad \to \quad \delta I = 8\pi \int d^2x \varphi (T\bar{T})_0, \tag{3.42}$$

which preserves the conformal symmetry of the undeformed CFT. The spurion doesn't have a physical set of conformal weights, but that's OK since at the end of the calculation we will demote it back to a constant.[8] Under a conformal transformation, the one-point function of any primary field $\mathcal{O}$ with weights $(h, \bar{h})$ transforms according to[9]

$$\langle \mathcal{O} \rangle_{\tau', \bar{\tau}'} = (c\tau + d)^h (c\bar{\tau} + d)^{\bar{h}} \langle \mathcal{O} \rangle_{\tau, \bar{\tau}}. \tag{3.43}$$

From this point of view, we see that the dimensionful deformation parameter transforms according to (3.39). The crucial point in this argument is that the $T\bar{T}$ operator has a fixed conformal dimension that does not receive any corrections in the deformed theory. This is not a generic feature of other deformations, with the exception of the X_s and related operators described in previous lectures, reasons why we expect modular invariance to be generically broken by deformations that break the conformal symmetry of the undeformed theory.

Now that we have some intuition as to why (3.39) may hold, let us show that this is indeed the case. Our strategy will be to expand the deformed energy (2.86) perturbatively in powers of the deformation parameter $\hat{\mu}$ such that

$$E_n(\hat{\mu}) = E_n - (E_n^2 - P_n^2)\hat{\mu} + 2E_n(E_n^2 - P_n^2)\hat{\mu}^2 + \mathcal{O}(\hat{\mu}^3), \tag{3.44}$$

where (E_n, P_n) denote the undeformed energy and angular momentum. As a result, the $T\bar{T}$ partition function (3.38) admits a perturbative expansion that we write as

$$Z(\tau, \bar{\tau}, \hat{\mu}) = \sum_{k=0}^{\infty} Z_k(\tau, \bar{\tau}) \hat{\mu}^k \tag{3.45}$$

[8] In a CFT, unitarity constrains the conformal weights of any state/operator to be positive or zero.

[9] A primary field $\mathcal{O}$ is in a highest weight representation of the Virasoro algebra. In particular, a generic field $\mathcal{O}$ is associated with a state $|\mathcal{O}\rangle$ that satisfies $L_0\mathcal{O} = h\mathcal{O}$ and $L_n\|O\rangle = 0$ for all $n > 0$. One exception is the vacuum $|0\rangle$ which satisfies $L_n|0\rangle = 0$ for all $n \geq -1$.

Each of the $Z_k(\tau, \bar{\tau})$ terms above can be thought as a thermal expectation value for powers of the energy and angular momentum,

$$Z_k(\tau, \bar{\tau}) = \text{Tr}\big(f_n^{(k)} e^{-\beta E_n + i\theta P_n}\big), \tag{3.46}$$

where the $f_n^{(k)}$ are functions of E_n and P_n determined by the perturbative expansion of the spectrum (3.44). The first few $f_n^{(k)}$ functions are given by

$$f_n^{(0)} = 1, \tag{3.47}$$

$$f_n^{(1)} = \beta(E_n^2 - P_n^2), \tag{3.48}$$

$$f_n^{(2)} = -2\beta E_n(E_n^2 - P_n^2) + \frac{\beta^2}{2}(E_n^2 - P_n^2)^2, \tag{3.49}$$

These expressions depend only on β, and not on θ, since the momentum P_n is unchanged by the deformation.

The crucial observation needed to prove modular invariance of the $T\bar{T}$ partition function is that each of the $Z_k(\tau, \bar{\tau})$ functions transform like a modular form of weight (k, k) under modular transformations, meaning that

$$Z_k(\gamma\tau, \gamma\bar{\tau}) = (c\tau + d)^k (c\tau + d)^k Z_k(\tau, \bar{\tau}). \tag{3.50}$$

This is obvious for $k = 0$, since in this case $Z_0(\tau, \bar{\tau}) = Z(\tau, \bar{\tau}; 0)$ is the undeformed partition function, which is invariant under modular transformations. In order to show that (3.50) holds for $k \geq 1$, it's convenient to relate each of the $Z_k(\tau, \bar{\tau})$ functions to the undeformed partition function $Z_0(\tau, \bar{\tau})$ in the following way. First, we note that each power of the energy and momentum in (3.46) can be obtained from the action of a derivative on the undeformed partition function, namely

$$\begin{aligned} \text{Tr}\big((E_n)^m e^{-\beta E_n + i\theta P_n}\big) &= \Big[\frac{1}{2\pi i}(\partial_\tau - \partial_{\bar{\tau}})\Big]^m Z_0(\tau, \bar{\tau}), \\ \text{Tr}\big((P_n)^m e^{-\beta E_n + i\theta P_n}\big) &= \Big[\frac{1}{2\pi i}(\partial_\tau + \partial_{\bar{\tau}})\Big]^m Z_0(\tau, \bar{\tau}). \end{aligned} \tag{3.51}$$

This means that each of the $f_n^{(k)}$ functions in $Z_k(\tau, \bar{\tau})$ can be written as a differential operator acting on $Z_0(\tau, \bar{\tau})$. For example, for $k = 0, 1, 2$ we have

$$f_n^{(0)} = 1, \tag{3.52}$$

$$f_n^{(1)} = \frac{2}{\pi}\text{Im}\,\tau\partial_\tau\partial_{\bar{\tau}}, \tag{3.53}$$

$$f_n^{(2)} = \frac{2}{\pi^2}\text{Im}\,\tau\partial_\tau^2\partial_{\bar{\tau}}^2 + \frac{2i}{\pi^2}\text{Im}\,\tau(\partial_\tau - \partial_{\bar{\tau}})\partial_\tau\partial_{\bar{\tau}}, \tag{3.54}$$

where $\text{Im}\,\tau = \beta/2\pi$.

Exercise 3.4: find the differential operator corresponding to $f_n^{(3)}$.

In order to prove (3.50) we need to understand how each of the $f_n^{(k)}$, written as a differential operator, transforms under modular transformations. The transformation of the imaginary part of τ is not difficult to deduce, and we find that it transforms as a modular form of weight $(-1,-1)$, namely

$$\mathrm{Im}\,\tau' = (c\tau+d)^{-1}(c\bar{\tau}+d)^{-1}\mathrm{Im}\,\tau, \tag{3.55}$$

where $\tau' = \gamma\tau = \frac{a\tau+b}{c\tau+d}$. On the other hand, the transformation of the ∂_τ and $\partial_{\bar{\tau}}$ derivatives is more subtle. By themselves, the derivatives transform as

$$\partial_{\tau'} = (c\tau+d)^2\partial_\tau, \qquad \partial_{\bar{\tau}'} = (c\bar{\tau}+d)^2\partial_{\bar{\tau}}. \tag{3.56}$$

However, when acting on a generic modular form, these derivatives don't transform *nicely* under modular transformations. The reason for this is that, generically, derivatives of modular forms are not themselves modular forms. For example, for a modular form $f_{k,\bar{k}}(\tau,\bar{\tau})$ of weight $(k,\bar{k})$ we find that

$$\partial_{\tau'} f_{k,\bar{k}}(\tau,\bar{\tau}) = (c\tau+d)^{k+2}(c\bar{\tau}+d)^{\bar{k}} f_{k,\bar{k}}(\tau,\bar{\tau}) + ck(c\tau+d)^{k+2}(c\bar{\tau}+d)^{\bar{k}} f_{k,\bar{k}}(\tau,\bar{\tau}), \tag{3.57}$$

which is not a modular form unless the second term vanishes, i.e. unless $k=0$. Nevertheless, it's possible to define covariant derivatives that transform nicely when acting on modular forms.

Let us introduce the following covariant derivatives

$$D_\tau^{(k)} := \partial_\tau - \frac{ik}{2\mathrm{Im}\,\tau}, \qquad D_{\bar{\tau}}^{(\bar{k})} := \partial_{\bar{\tau}} + \frac{i\bar{k}}{2\mathrm{Im}\,\tau}. \tag{3.58}$$

The covariant derivatives $D_\tau^{(k)}$ and $D_{\bar{\tau}}^{(\bar{k})}$ map a modular form of weight $(k,\bar{k})$ to a modular form of weight $(k+2,\bar{k})$ and $(k,\bar{k}+2)$, respectively. In other words, we have

$$\begin{aligned} D_{\tau'}^{(k)} f_{k,\bar{k}}(\tau',\bar{\tau}') &= (c\tau+d)^{k+2}(c\bar{\tau}+d)^{\bar{k}} f_{k,\bar{k}}(\tau,\bar{\tau}), \\ D_{\bar{\tau}'}^{(\bar{k})} f_{k,\bar{k}}(\tau',\bar{\tau}') &= (c\tau+d)^{k}(c\bar{\tau}+d)^{\bar{k}+2} f_{k,\bar{k}}(\tau,\bar{\tau}). \end{aligned} \tag{3.59}$$

Exercise 3.5: prove the transformation law of the covariant derivatives $D_\tau^{(k)}$ and $D_{\bar{\tau}}^{(\bar{k})}$ given in (3.59).

Note that the covariant derivatives (3.58) don't commute when acting on a modular form of weight $(k,\bar{k})$ since

$$\left[D_\tau^{(k)}, D_{\bar{\tau}}^{(\bar{k})}\right] = \frac{k-\bar{k}}{4(\mathrm{Im}\,\tau)^2}. \tag{3.60}$$

As a result, the order in which $D_\tau^{(k)}$ and $D_{\bar\tau}^{(\bar k)}$ act on a modular form is important.

Exercise 3.6: does the order of the $D_\tau^{(k)}$ derivatives (or the $D_{\bar\tau}^{(\bar k)}$ derivatives) matter?

In terms of the covariant derivatives (3.58), each of the $f_n^{(k)}$ terms we have considered in this lecture can be written as

$$f_n^{(0)} = 1, \tag{3.61}$$

$$f_n^{(1)} = \frac{2}{\pi}\mathrm{Im}\,\tau D_\tau^{(0)} D_{\bar\tau}^{(0)}, \tag{3.62}$$

$$f_n^{(2)} = \frac{1}{2}\left(\frac{2}{\pi}\right)^2 (\mathrm{Im}\,\tau)^2 D_\tau^{(2)} D_\tau^{(0)} D_{\bar\tau}^{(2)} D_{\bar\tau}^{(0)}. \tag{3.63}$$

The $f_n^{(k)}$ continue to take this form for larger values of k such that, for $k = 0, 1, 2, \dots$, we find

$$f_n^{(k)} = \frac{1}{k!}\left(\frac{2}{\pi}\right)^k (\mathrm{Im}\,\tau)^k D_\tau^{(2k-2)} D_\tau^{(2k-4)} \dots D_\tau^{(2)} D_\tau^{(0)}\, D_{\bar\tau}^{(2k-2)} D_{\bar\tau}^{(2k-4)} \dots D_{\bar\tau}^{(2)} D_{\bar\tau}^{(0)}. \tag{3.64}$$

Exercise 3.7: using the results of exercise 3.2, confirm that the differential operator corresponding to $f_n^{(3)}$ can indeed be written as (3.64).

The fact that (3.64) features only even superscripts follows from the fact that all the derivatives act on $Z_0(\tau,\bar\tau)$, which is a modular form of weight $(0,0)$, and the fact that each derivative increases the weight by 2. In addition, note that since the $D_\tau^{(k)}$ and $D_{\bar\tau}^{(\bar k)}$ derivatives don't commute, the derivatives must be evaluated in the order in which they appear in (3.64).

The transformation law of $\mathrm{Im}\,\tau$ (3.55) and the covariant derivatives $D_\tau^{(k)}$ and $D_{\bar\tau}^{(\bar k)}$ (3.59) implies that $Z_k(\tau,\bar\tau)$ is a modular form of weight (k,k), at least for small values of k,

$$Z_k(\gamma\tau,\gamma\bar\tau) = (c\tau+d)^k(c\bar\tau+d)^k Z_k(\tau,\bar\tau). \tag{3.65}$$

We will now use Burger's equation (2.79) to show that (3.65) extends to all k. Let us consider the version of Burger's equation written in terms of dimensionless variables (2.84), which we reproduce here for convenience

$$\partial_{\hat\mu}\big(RE_n(\hat\mu) + \hat\mu R^2 E_n(\hat\mu)^2 - \hat\mu(RP_n)^2\big) = 0. \tag{3.66}$$

Since this combination of terms is zero for every state in a $T\bar T$ deformed CFT, its insertion in the deformed partition function yields

$$-\pi^2 \mathrm{Im}\,\tau\,\mathrm{Tr}\big[\big(\partial_{\hat\mu}E_n(\hat\mu) + 2\hat\mu E_n(\hat\mu)\partial_{\hat\mu}E_n(\hat\mu) + E_n(\hat\mu)^2 - (P_n)^2\big) q^{E_L(\hat\mu)}\bar q^{E_R(\hat\mu)}\big] = 0, \tag{3.67}$$

where we have set $R = 1$ and the factor of $-\pi^2\mathrm{Im}\,\tau$ has been added for convenience. Using

(3.51), we can write this identity as a differential equation for the partition function,

$$\left[\frac{\pi}{2}\partial_{\hat{\mu}} - \mathrm{Im}\,\tau\partial_\tau\partial_{\bar{\tau}} - \frac{i}{2}\left(\partial_\tau - \partial_{\bar{\tau}} + \frac{i}{\mathrm{Im}\,\tau}\right)\hat{\mu}\partial_{\hat{\mu}}\right]Z(\tau,\bar{\tau};\hat{\mu}) = 0. \tag{3.68}$$

Applied to the perturbative expansion of the partition function (3.45), this differential equation implies that

$$\sum_{k=0}^{\infty}\left\{\frac{\pi k}{2}\hat{\mu}^{k-1} - \left[\mathrm{Im}\,\tau\left(\partial_\tau - \frac{ik}{2\mathrm{Im}\,\tau}\right)\left(\partial_{\bar{\tau}} + \frac{ik}{2\mathrm{Im}\,\tau}\right) - \frac{k(k+1)}{4\mathrm{Im}\,\tau}\right]\hat{\mu}^k\right\}Z_k(\tau,\bar{\tau}) = 0. \tag{3.69}$$

As a result, the partition function $Z_{k+1}(\tau,\bar{\tau})$ is related to $Z_k(\tau,\bar{\tau})$ by

$$Z_{k+1}(\tau,\bar{\tau}) = \frac{2}{\pi(k+1)}\left(\underbrace{\mathrm{Im}\,\tau D_\tau^{(k)}D_{\bar{\tau}}^{(k)}}_{(1,1)} - \underbrace{\frac{k(k+1)}{4\mathrm{Im}\,\tau}}_{(1,1)}\right)\underbrace{Z_k(\tau,\bar{\tau})}_{(k,k)}, \tag{3.70}$$

where we used the definition of the covariant derivatives (3.58).

The recursive relation (3.70) shows that if $Z_k(\tau,\bar{\tau})$ has weight (k,k), which we have established for small values of k, then $Z_{k+1}(\tau,\bar{\tau})$ has weight $(k+1,k+1)$. By induction, it follows that $Z_k(\tau,\bar{\tau})$ is a modular form of weight (k,k) for all k, thereby establishing (3.65). Consequently, each of the terms in the perturbative expansion of the partition function (3.45) is modular invariant, provided that $\hat{\mu}$ has weight $(-1,-1)$, namely

$$\hat{\mu}' = (c\tau+d)^{-1}(c\bar{\tau}+d)^{-1}\hat{\mu}. \tag{3.71}$$

This is precisely the transformation of the dimensionless deformation parameter proposed earlier in (3.39). Thus, we have proved that the partition function of any $T\bar{T}$-deformed CFT is modular invariant provided that (3.71) holds.

We have seen that the spectrum of $T\bar{T}$-deformed CFTs implies a modular invariant torus partition function. We argued that this result is reasonable due to the fact the $T\bar{T}$ operator has a fixed scaling dimension at any point during the deformation. A natural question to ask is whether the converse is true, namely, whether modular invariance (3.39) implies the $T\bar{T}$ spectrum (2.86). It turns out that, while it is possible to show that this is indeed the case (up to possible nonperturbative corrections), the assumptions necessary to establish this result are very constraining, and lead precisely to the kind of $T\bar{T}$-deformed CFTs we have been considering in these lectures. Nevertheless, there is a larger class of theories that admit the $T\bar{T}$ spectrum (2.86) and additional states satisfying closely related formulae, whose partition functions also satisfy (3.39). As a result, we find that modular invariance is not only compatible with the spectrum (2.86) but with more general formulae that include sums of square roots, as we will see in future lectures.

References

[1] A. Giveon, N. Itzhaki and D. Kutasov, $\mathrm{T}\overline{\mathrm{T}}$ *and LST*, *JHEP* **07** (2017) 122 [1701.05576].

[2] S. Datta and Y. Jiang, $T\bar{T}$ *deformed partition functions*, *JHEP* **08** (2018) 106 [1806.07426].

[3] Y. Jiang, *Lectures on solvable irrelevant deformations of 2d quantum field theory*, 1904.13376.