Lecture 6

In this lecture we explore the consequences of modular invariance in $T\bar{T}$-deformed CFTs. We will show that $T\bar{T}$-deformed CFTs have a universal torus partition function and a universal density of high energy states. In particular, we will see that the asymptotic density of states is compatible with the existence of a maximum temperature and indicates that $T\bar{T}$-deformed CFTs remain nonlocal at arbitrarily high energies.

3.3 Maximum temperature

Let us now study one of the implications of modular invariance in $T\bar{T}$-deformed CFTs. We first note that the change of the dimensionless deformation parameter (3.71) under modular transformations implies that $T\bar{T}$-deformed CFTs have a maximum temperature. This follows from the fact, as discussed previously, that the deformation parameter is bounded from above by

$$\mu \leq \frac{3}{c}, \tag{3.72}$$

where we have set $R = 1$ with respect to (2.93). Consequently, under a modular $\mathcal{S}$ transformation $\tau \to -1/\tau$ we find that

$$\hat{\mu} \to \hat{\mu}' = \frac{\hat{\mu}}{|\tau|^2} \qquad \Longrightarrow \qquad |\tau|^2 \geq \frac{c\mu}{3}. \tag{3.73}$$

If we turn off the angular potential, this implies that the inverse temperature β is bounded from below by

$$\beta^2 \geq 4\pi^2 \frac{c\mu}{3}. \tag{3.74}$$

Hence $T\bar{T}$-deformed CFTs have a maximum temperature.

The maximum temperature of $T\bar{T}$-deformed CFTs is reminiscent of string theory, which also features a maximum *Hagedorn* temperature. The latter is a consequence of a density of states that grows too fast at high energies, being exponential in the energy

$$\rho_H(E) \sim e^{2\pi E}. \tag{3.75}$$

This so-called Hagedorn density of states is a signature of non-locality. For a local theory like a CFT, the growth of high energy states is much slower, and as we have seen, given by the Cardy formula. Indeed, if we turn off the momentum, the asymptotic density of states in a CFT grows as $\rho(E) \sim e^{2\pi\sqrt{\frac{c}{3}RE}}$, which is much slower than the Hagedorn growth (3.79). We will now show that $T\bar{T}$-deformed CFTs also feature a Hagedorn growth of high energy states, a fact that is closely related to the existence of a maximum temperature.

3.4 Asymptotic density of states

In analogy with the analysis carried out in the previous lecture, modular invariance of the $T\bar{T}$ partition function implies a universal asymptotic density of states. In order to see this, we begin by noting that the energy of any state in a $T\bar{T}$-deformed CFT cannot change sign under the deformation. This is evident from the square-root structure of the spectrum (2.86) (see also figure 6). In particular, states that had negative energy before the deformation, continue to have negative energy after the deformation. In addition, we note that there is no level crossing in the spectrum of $T\bar{T}$-deformed CFTs. This means that if the states in the undeformed theory satisfy

$$E_1(0) < E_2(0) < E_3(0) < \dots, \tag{3.76}$$

then after the deformation, the deformed states continue to satisfy this hierarchy, that is[10]

$$E_1(\mu) < E_2(\mu) < E_3(\mu) < \dots \tag{3.77}$$

These two statements imply that the ground state or vacuum energy of a $T\bar{T}$-deformed CFT is negative and given by the $T\bar{T}$ transformation of the CFT vacuum such that

$$E_{\text{vac}}(\mu) = -\frac{1}{2\mu}\left(1 - \sqrt{1 - \frac{c\mu}{3}}\right). \tag{3.78}$$

The modular invariance of the $T\bar{T}$ partition function (3.39) implies that the torus partition function is dominated by the vacuum at high (but not too high) temperatures $|\tau| < 1$, namely

$$Z(\tau, \bar{\tau}; \hat{\mu}) = Z\left(-\frac{1}{\tau}, -\frac{1}{\bar{\tau}}; \frac{\hat{\mu}}{\tau\bar{\tau}}\right) \approx e^{\pi i E_{\text{vac}}\left(\frac{\mu}{\tau\bar{\tau}}\right)\frac{\tau-\bar{\tau}}{\tau\bar{\tau}}}. \tag{3.79}$$

This is analogous to what we found for the undeformed CFT (3.9) except that now, the vacuum energy changes under modular transformations due to its dependence on the deformation parameter. This feature of $T\bar{T}$-deformed CFTs is important in determining the density of high energy states as we'll soon find.

Let us now extract the density of high energy states from (3.79). Before we do this it's convenient to perform, once again, the analytic continuation of $(\tau, \bar{\tau})$ given in (3.10). In addition, we assume that the density of states is a continuous variable such that, in analogy with the CFT analysis carried out in the previous section, the partition function can be written as

$$Z(\tau, \bar{\tau}; \hat{\mu}) = \int dE_L \int dE_R\, \rho(E_L, E_R) e^{-2\pi\gamma E_L(\mu) - 2\pi\bar{\gamma} E_R(\mu)}. \tag{3.80}$$

The density of high energy states can then be obtained from the inverse Laplace transform of

[10] Note that this holds for any value of μ, whether it's positive or negative.

(3.79) such that

$$\rho(E_L, E_R) = \int d\gamma \int d\bar{\gamma}\, e^{2\pi\gamma E_L(\mu)+2\pi\bar{\gamma}E_R(\mu)-\pi E_{\rm vac}(\frac{\mu}{\gamma\bar{\gamma}})\frac{\gamma+\bar{\gamma}}{\gamma\bar{\gamma}}}. \tag{3.81}$$

Using the saddle-point approximation it's possible to show that the saddle is located at

$$\begin{aligned}\gamma^* &= \frac{1}{2}\left(\sqrt{\frac{\frac{c}{6}(1+2\mu E_R(\mu))}{E_L(\mu)}} + 2\mu\sqrt{\frac{\frac{c}{6}E_R(\mu)}{1+2\mu E_L(\mu)}}\right),\\ \bar{\gamma}^* &= \frac{1}{2}\left(\sqrt{\frac{\frac{c}{6}(1+2\mu E_L(\mu))}{E_R(\mu)}} + 2\mu\sqrt{\frac{\frac{c}{6}E_L(\mu)}{1+2\mu E_R(\mu)}}\right).\end{aligned} \tag{3.82}$$

In particular, we can verify that in the limit $\mu \to 0$, the $T\bar{T}$ saddle reduces to the CFT one given in (3.14). The asymptotic density of states obtained from (3.81) is therefore given by

$$S(\mu) = \log\rho(E_L, E_R) = 2\pi\sqrt{\frac{c}{6}E_L(\mu)\Big(1+2\mu E_R(\mu)\Big)} + 2\pi\sqrt{\frac{c}{6}E_R(\mu)\Big(1+2\mu E_L(\mu)\Big)}. \tag{3.83}$$

Exercise 3.8: consider the case $E_L = E_R = 1/2E$ and verify that (3.82) are indeed saddles of (3.81). Compute the asymptotic density of states in this case, including the logarithmic corrections, and compare the logarithmic corrections to the ones you computed in exercise 3.1.

The asymptotic density of states (3.83) is valid at high temperatures provided the temperature is compatible with the asymptotic bound (3.73). This requires the energies to satisfy

$$E_L(\mu) \gg 1, \qquad E_R(\mu) \gg 1, \tag{3.84}$$

which is the Cardy regime (3.16) written in terms of the energies instead of the conformal weights. Note that, while the temperature is bounded from above by (3.73), the energies can become arbitrarily high. In particular, we find that at high energies the density of states grows much faster than that of the undeformed CFT and is Hagedorn-like (exponential in the energy)

$$\rho(E) \approx e^{2\pi\sqrt{\frac{c\mu}{3}}E(\mu)}, \qquad E(\mu) \gg 1. \tag{3.85}$$

This is a clear indication that the $T\bar{T}$ deformation drastically changes the UV behavior of the theory. In particular, the bound on the deformation parameter (3.72) implies that the density of states (3.85) is bounded from above by Hagedorn's

$$\rho(E) \le \rho_H(E). \tag{3.86}$$

As a consistency check, we verify that the Hagedorn-like growth (3.85) implies a maximum Hagedorn temperature. This can be seen by turning off the angular momentum, in which case the partition function becomes

$$Z(\tau,\bar{\tau};\hat{\mu}) = \int dE\,\rho(E)e^{-\beta E(\mu)} \approx \int dE\, e^{(2\pi\sqrt{\frac{c\mu}{3}}-\beta)E(\mu)}, \tag{3.87}$$

where we approximated the density of states $c(E)$ by a continuous function $\rho(E)$, the latter of which is approximated by (3.85) at high energies. Thus, we see that the partition function becomes ill defined whenever (3.74) is violated, which confirms the existence of a maximum temperature.

Finally, we note that the combination of energies appearing in the square roots of (3.83) is special. It's not difficult to show that Burger's equation (2.79) can be alternatively written as

$$E_L(0) = E_L(\mu) + 2\mu E_L(\mu)E_R(\mu), \qquad E_R(0) = E_R(\mu) + 2\mu E_L(\mu)E_R(\mu). \tag{3.88}$$

The difference of these expressions is independent of μ since the angular momentum is unchanged by the deformation. On the other hand, it's not difficult to verify that the sum of these equations yields the dimensionful version of the quadratic equation (2.85). Using (3.88), the log of the density of states (3.83) can be written as

$$S(\mu) = 2\pi\sqrt{\frac{c}{6}\underbrace{E_L(0)}_{h-\frac{c}{24}}} + 2\pi\sqrt{\frac{c}{6}\underbrace{E_R(0)}_{\bar{h}-\frac{c}{24}}} = S(0), \tag{3.89}$$

which is nothing but the undeformed Cardy formula (3.15) written in terms of the energies. This result is not surprising. It's telling us that the asymptotic density of states of a $T\bar{T}$-deformed CFT, i.e. the (log of the) number of states in a small window centered around $(E_L(\mu), E_R(\mu))$, is unchanged by the deformation. In other words, this number is given by the number of states in a small window around $(E_L(0), E_R(0))$ in the undeformed CFT. Since there is no level crossing in the spectrum of $T\bar{T}$-deformed CFTs, i.e. since the hierarchies (3.76) and (3.77) are preserved, so is the number of states at a given energy.

The results in this section tie in nicely with comments made in previous lectures about the nature of the $T\bar{T}$ deformation. In particular, the asymptotic Hagedorn-like density of states (3.83) shows that a $T\bar{T}$-deformed CFT doesn't flow to a UV fixed point, i.e. it's a nonlocal QFT distinct from a CFT. Relatedly, the $T\bar{T}$ deformation should not be interpreted as an RG flow and in particular, it doesn't add any new degrees of freedom, as shown in (3.89).

3.5 Partition function at large c

In previous lectures we found that the partition function of sparse two-dimensional CFTs is universal in the large-c limit: it's dominated by the vacuum at low temperatures and by its $\mathcal{S}$ modular image at high temperatures. This result relied only on modular invariance, a sparse density of states, and the existence of a ground state. We have seen that these properties are preserved by the $T\bar{T}$ deformation. Hence it's natural to ask if $T\bar{T}$-deformed CFTs also feature a universal partition function when the central charge of the undeformed CFT is large. We will now show that this is indeed the case.

Before we begin it's important to define what we mean by large-c in the context of $T\bar{T}$. There are two subtleties we need to address. The first one is that the $T\bar{T}$ deformation breaks the conformal symmetries of a CFT and preserves only its Poincaré subgroup, i.e. Lorentz transformations and translations. Hence, when we speak of the central charge, which may be thought to be an intrinsic CFT quantity, we are talking about the central charge of the *undeformed* theory. One way to determine the central charge in the deformed theory is through the Weyl anomaly. Indeed, although the conformal symmetry is broken by the $T\bar{T}$ operator, the trace of the stress tensor in a $T\bar{T}$-deformed CFT is universal and still sensitive to the Weyl anomaly so that

$$\langle T^{\mu}_{\mu}\rangle = -\frac{c}{24\pi}R^{(2)} + 2\pi\mu\langle T\bar{T}\rangle, \tag{3.90}$$

where $R^{(2)}$ is the Ricci scalar. When $\mu \to 0$ this expression reduces to the standard form of the Weyl anomaly.

The second subtletly comes from the fact that the deformation parameter is bounded from above by (2.93), which we reproduce here for convenience

$$\mu \leq \frac{3}{c}. \tag{3.91}$$

As a result, the semiclassical or large-c limit of a $T\bar{T}$-deformed CFT requires us to scale both the central charge and the deformation parameter such that

$$c \gg 1, \qquad c\mu \text{ fixed}. \tag{3.92}$$

Thus, whenever we refer to the large-c limit of a $T\bar{T}$-deformed CFT we have in mind (3.92).

Let us begin by assuming that $\beta > 2\pi$ and $\theta = 0$. Next, we note that states with zero energy are fixed points of the deformation, in the sense that the energy doesn't change under the deformation, i.e. $E(0) = 0 \implies E(\mu) = 0$. Furthermore, we recall that the energy of any state in a $T\bar{T}$-deformed CFTs preserves the sign of the energy before the deformation. As a result, it's natural to extend the definition of light and heavy states given in (3.19) to the context of $T\bar{T}$-deformed CFTs. We thus have

$$\begin{aligned} \text{Light states: } E(\mu) &\leq \epsilon, \\ \text{Heavy states: } E(\mu) &> \epsilon. \end{aligned} \tag{3.93}$$

except that the energy is now measured with respect to the energy of the vacuum (3.78), which is negative and μ-dependent. The $T\bar{T}$ partition function can then be split into contributions of light and heavy states, which are defined as in (3.20). The only novelty in the present case is that the deformation parameter changes under modular transformations such that

$$
\begin{aligned}
Z[L] &:= \mathrm{Tr}_L(e^{-\beta E(\mu)}), \qquad Z'[L] := \mathrm{Tr}_L(e^{-\beta' E(\mu')}),\\
Z[H] &:= \mathrm{Tr}_H(e^{-\beta E(\mu)}), \qquad Z'[H] := \mathrm{Tr}_H(e^{-\beta' E(\mu')}),
\end{aligned}
\tag{3.94}
$$

where $\beta' := 4\pi^2/\beta$ and $\mu' := 4\pi^2/\beta^2$. As before, modular invariance implies that that these partition functions are related according to

$$
Z(\beta;\hat{\mu}) = Z[L] + Z[H] = Z'[L] + Z'[H] = Z(\beta';\hat{\mu}'). \tag{3.95}
$$

When $\beta > 2\pi$, it's not difficult to show that $E(\mu') \geq E(\mu)$, with the equality being saturated only for the fixed point with $E(\mu) = 0$. This follows from the fact that the energy decreases monotonically as we increase the deformation and that $\mu' < \mu$. As a result, we find that the heavy states $E(\mu) > \epsilon$ satisfy

$$
\beta' E(\mu') - \beta E(\mu) < 0. \tag{3.96}
$$

This is the same relationship satisfied by the heavy states in the undeformed theory, see (3.22). As a result, we can repeat the same steps relating the heavy and light partition functions in the undeformed CFT. We thus find that the $T\bar{T}$ partition function is bounded from above and below by the light state partition function,

$$
\log Z[L] < \log Z(\beta;\hat{\mu}) < \log Z[L] - \log(1-\tilde{\alpha}). \tag{3.97}
$$

where $0 < \tilde{\alpha} < 1$ is a number of $\mathcal{O}(c^0)$ at large c. Consequently, the partition function of any $T\bar{T}$-deformed CFT with a large undeformed central charge is dominated by the light states at low temperatures, namely

$$
\log Z(\beta;\hat{\mu}) \approx \log \mathrm{Tr}_L\big(e^{-\beta E(\hat{\mu})}\big). \tag{3.98}
$$

The light state dominance of the partition function (3.98) can be made universal, i.e. independent of the distribution of light states, by requiring the spectrum to be sparse enough so that only the vacuum dominates. The sparseness condition turns out to take the same form as the one imposed on the undeformed CFT in (3.29). As a result, the sparseness condition

$$
\rho(E) < e^{2\pi(E(\mu) - E_{\mathrm{vac}}(\mu))}, \qquad E \leq \epsilon, \tag{3.99}
$$

guarantees that the partition function is universal at low temperatures and large-c (the latter condition being necessary to suppress the contribution of the light states),

$$
Z(\beta;\hat{\mu}) \approx e^{-\beta E_{\mathrm{vac}}(\mu)}, \qquad \beta > 2\pi. \tag{3.100}
$$

At high temperatures the partition function is dominated by the modular image of the vacuum, which yields a novel dependence on the inverse temperature β due to the transformation of the deformation parameter,

$$Z(\beta;\hat{\mu}) \approx e^{-\frac{4\pi^2}{\beta}E_{\text{vac}}(\frac{4\pi^2\mu}{\beta^2})}, \qquad \beta < 2\pi. \tag{3.101}$$

Since the deformation parameter transforms under the $\mathcal{S}$ modular transformation, it's important to check that vacuum dominance continues to hold in the $\beta < 2\pi$ case. We will argue that this is indeed the case below, after we consider the more general case with $\theta \neq 0$.

In analogy with the analysis of the undeformed partition function, we can generalize these results to include an angular potential θ. The appropriate sparseness condition that guarantees universality of the partition function is the natural generalization of (3.32), namely

$$\rho(E_L, E_R) \leq e^{4\pi\sqrt{(E_L(\mu)-\frac{1}{2}E_{\text{vac}}(\mu))(E_R(\mu)-\frac{1}{2}E_{\text{vac}}(\mu))}}, \qquad E_L(\mu) \leq \epsilon \quad \text{or} \quad E_R(\mu) \leq \epsilon. \tag{3.102}$$

Thus, the torus partition function of sparse $T\bar{T}$-deformed CFTs is universal at large c and given at low and high temperatures by

$$Z(\tau,\bar{\tau};\hat{\mu}) \approx \begin{cases} e^{\pi i(\tau-\bar{\tau})E_{\text{vac}}(\mu)}, & |\tau| > 1, \\ e^{-\pi i\left(\frac{1}{\tau}-\frac{1}{\bar{\tau}}\right)E_{\text{vac}}(\frac{\mu}{\tau\bar{\tau}})}, & |\tau| < 1. \end{cases} \tag{3.103}$$

The partition function (3.103) is another remarkable consequences of modular invariance and another way in which the $T\bar{T}$ deformation is universal. In the second part of the course, we will show that this partition function can be reproduced holographically from three-dimensional gravity with a negative cosmological constant.

The universality of the partition function (3.103) implies a universal density of high energy states. In the canonical ensemble, the log of the density of states at high temperatures $|\tau| < 1$ is simply given by

$$\begin{aligned} S(\tau,\bar{\tau};\mu) &= (1-\tau\partial_\tau-\bar{\tau}\partial_{\bar{\tau}})\log Z(\tau,\bar{\tau};\hat{\mu}), \\ &= \frac{i\pi c}{6}\Big(1-\frac{\mu c}{3\tau\bar{\tau}}\Big)^{-1/2}\Big(\frac{1}{\tau}-\frac{1}{\bar{\tau}}\Big). \end{aligned} \tag{3.104}$$

This formula takes a similar form to Cardy's formula in the canonical ensemble (3.34) except for the additional μ-dependent factor which originates from the energy of the vacuum (3.78).

Interestingly, the sparseness conditions on the light states (3.99) and (3.102) take the same functional form as those of the undeformed CFT in (3.29) and (3.32). Crucially, the former are written in terms of the deformed energies, while the latter are written in terms of the undeformed ones. This means that the $T\bar{T}$ sparseness conditions cannot be obtained by rewriting the CFT sparseness conditions in terms of the $T\bar{T}$ spectrum via (2.86). In other words, the numerical value of the $T\bar{T}$ and CFT sparseness conditions differ from each other. This is in sharp contrast with the density of high energy states described earlier, which takes the same numerical value before and after the deformation, but differs in its functional form: it scales as $e^{\sqrt{E(0)}}$ before the deformation but as $e^{E(\mu)}$ after the deformation.

It's important to note that the terms on the RHS of the sparseness conditions (3.99) and (3.102) grow monotonically with μ. This means, in particular, that the $T\bar{T}$ sparseness conditions are weaker than the CFT ones. As a result, a sparse CFT remains sparse after the $T\bar{T}$ deformation but the converse is not necessarily true. Furthermore, we note that in order to establish vacuum dominance in the $|\tau| < 1$ region, it is necessary to revisit the derivation of vacuum dominance at $\mu' = \frac{\mu}{\tau\bar{\tau}} > \mu$. Since the sparseness conditions become weaker the larger μ is, vacuum dominance at μ' follows automatically from vacuum dominance at μ. Consequently, we only needed to consider the case $|\tau| > 1$ when establishing vacuum dominance of the partition function at low and high temperatures.

At high temperatures, the average (thermal expectation values of the) energies are given by

$$\begin{aligned} E_L(\mu) &\coloneqq \langle E_L(\mu)\rangle_{\tau,\bar{\tau}} = \frac{1}{2\pi i}\partial_\tau \ln Z(\tau,\bar{\tau};\mu) = \frac{1}{2}\left(\frac{1}{\tau^2} + \frac{\tau-\bar{\tau}}{|\tau|^2}\partial_\tau\right)E_{\rm vac}\left(\frac{\mu}{\tau\bar{\tau}}\right), \\ E_R(\mu) &\coloneqq \langle E_R(\mu)\rangle_{\tau,\bar{\tau}} = -\frac{1}{2\pi i}\partial_{\bar{\tau}} \ln Z(\tau,\bar{\tau};\mu) = \frac{1}{2}\left(\frac{1}{\bar{\tau}^2} - \frac{\tau-\bar{\tau}}{|\tau|^2}\partial_{\bar{\tau}}\right)E_{\rm vac}\left(\frac{\mu}{\tau\bar{\tau}}\right). \end{aligned} \tag{3.105}$$

After some algebra, these expressions can be shown to be equivalent to

$$\begin{aligned} \tau &= \frac{i}{2}\left(\sqrt{\frac{\frac{c}{6}(1+2\mu E_R(\mu))}{E_L(\mu)}} + 2\mu\sqrt{\frac{\frac{c}{6}E_R(\mu)}{1+2\mu E_L(\mu)}}\right), \\ \bar{\tau} &= -\frac{i}{2}\left(\sqrt{\frac{\frac{c}{6}(1+2\mu E_L(\mu))}{E_R(\mu)}} + 2\mu\sqrt{\frac{\frac{c}{6}E_L(\mu)}{1+2\mu E_R(\mu)}}\right). \end{aligned} \tag{3.106}$$

Not surprisingly, these are the same equations obtained from the saddle-point approximation in the previous section, see (3.82). Plugging these expressions into the entropy (3.104) yields the microcanonical density of states we found earlier, namely

$$S(\mu) = \log\rho(E_L,E_R) = 2\pi\sqrt{\frac{c}{6}E_L(\mu)\Big(1+2\mu E_R(\mu)\Big)} + 2\pi\sqrt{\frac{c}{6}E_R(\mu)\Big(1+2\mu E_L(\mu)\Big)}. \tag{3.107}$$

Note that this derivation of the asymptotic density of states is valid at high temperatures where $|\tau|^2 < 1$. In terms of the energies, this constraint translates to

$$\frac{E_L(\mu)E_R(\mu)}{1+2\mu\big(E_L(\mu)+E_R(\mu)\big)} > \frac{E_{\rm vac}(\mu)^2}{4\big(1+2\mu E_{\rm vac}(\mu)\big)}. \tag{3.108}$$

This condition tells us that the energies must be greater than the central charge up to corrections that depend on $c\mu$ which is fixed in the large-c limit according to (3.92),

$$E_L(\mu)E_R(\mu) > \frac{c^2}{24^2}\big(1+\mathcal{O}(c\mu)\big), \tag{3.109}$$

thereby extending the regime of validity of (3.107) beyond the Cardy regime (3.84).

Exercise 3.9: let $E_L(\mu) = E_R(\mu)$ in (3.106), i.e. turn off the angular potential, and let $\tau = i\beta(\mu)/2\pi$. Use the results of exercise 3.1 to find a relationship between the deformed inverse temperature $\beta(\mu)$ and the undeformed one $\beta(0)$. What happens when $\beta(\mu)$ reaches saturates the bound (3.74)? Show that the asymptotic density of states (3.104) reduces to (3.34) when written in terms of the undeformed temperatures.

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