Lecture 7

In this lecture we will consider the symmetric product orbifold of a CFT. First we will define and motivate the study of these theories from the point of view of holography. We will then describe in detail the untwisted sector of the theory and its contribution to the torus partition function.

4 Symmetric product orbifolds

Let us begin by describing the symmetric product orbifold (or symmetric orbifold for short) of a two-dimensional CFT. Symmetric orbifolds are defined by taking N copies of a *seed* CFT $\mathcal{M}$ and symmetrizing the result (see figure 10)

$$\mathrm{Sym}^N\mathcal{M} := \frac{\mathcal{M}^N}{S_N}, \tag{4.1}$$

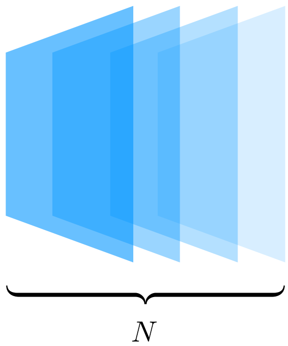

Figure 9: The symmetric product orbifold $\mathrm{Sym}^N\mathcal{M}$ consists of N symmetrized copies of $\mathcal{M}$.

This means, in particular, that the Hilbert space of a symmetric orbifold consists only of states that are invariant under the action of the symmetric group S_N. Symmetric orbifolds have several properties that make them interesting from a holographic point of view. In particular,

(*i*) symmetric orbifolds have a central charge that scales linearly with N. If we denote the central charge of the seed CFT $\mathcal{M}$ by c_0, then the central charge of $\mathrm{Sym}^N\mathcal{M}$ is

$$c = Nc_0. \tag{4.2}$$

Hence, as $N \to \infty$, we obtain a theory with a large central charge which, as we have discussed previously, is a necessary feature of a holographic CFT.

(*ii*) In addition, symmetric orbifolds have a universal torus partition function in the large-c limit which matches, as we will see, the partition function of three-dimensional Einstein gravity with a negative cosmological constant.

(*iii*) Finally, the correlation function of $\mathrm{Sym}^N\mathcal{M}$ factorize in the large-N limit, meaning that its two, three, and four-point functions scale with N as

$$\langle \mathcal{O}_i\mathcal{O}_i\rangle \sim N^0, \tag{4.3}$$

$$\langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\rangle \sim \frac{1}{N^{1/2}}, \tag{4.4}$$

$$\langle \mathcal{O}_i\mathcal{O}_j\mathcal{O}_k\mathcal{O}_l\rangle \sim \frac{1}{N}, \tag{4.5}$$

which is the same behavior found in gravitational theories in the semiclassical limit using the holographic dictionary $G_n \sim 1/c$.

Exercise 4.1: find the G_N-scaling of tree-level amplitudes with two, three, and four external vertices using the perturbative gravitational action of Einstein gravity in three dimensions (the analog of (1.1)). Confirm that these reproduce the large-N behavior of the two, three, and four-point functions using the holographic dictionary.

However, symmetric orbifolds also have holographically undesirable properties, including an infinite tower of massless higher spin currents, and as we will describe later on, a Hagedorn growth of low energy states in the large-N limit. Neither of these are properties of gravity in the semiclassical limit. For this reason, symmetric orbifolds cannot describe semiclassical gravity but are still relevant, as they may lie on the same moduli space as a holographic CFT. An example of this that will be relevant later on is the following. The symmetric orbifold of T^4, i.e. of the supersymmetric theory consisting of 4 free bosons and 4 free fermions, is dual to the tensionless limit of string theory on $\mathrm{AdS}_3 \times S^3 \times T^4$ supported by NS-NS flux (by an antisymmetric rank-2 field). An exactly marginal deformation of the symmetric orbifold is expected to remove the massless higher spin currents from the spectrum and deform the theory to the point where it meets all of the necessary conditions to be holographically dual to semiclassical gravity in the aforementioned background.

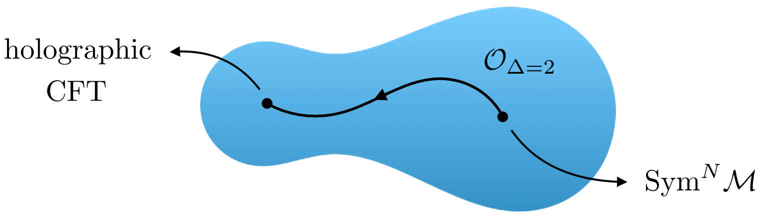

Figure 10: The symmetric product orbifold $\mathrm{Sym}^N\mathcal{M}$ consists of N symmetrized copies of $\mathcal{M}$.

4.1 Partition function

Let us now consider the torus partition function of $\mathrm{Sym}^N\mathcal{M}$, which we will denote by

$$Z_N(\tau,\bar{\tau}) = \mathrm{Tr}\,(q^{h-c/24}\bar{q}^{\bar{h}-c/24}). \tag{4.6}$$

The partition function receives contributions from two types of states, so-called untwisted and twisted states. The first type of states are a result of taking the product of N copies of the seed CFT $\mathcal{M}$. Whereas the second type of states originate from the symmetrization procedure. Let us first consider the contribution of the untwisted states.

Untwisted sector

The untwisted sector of $\text{Sym}^N\mathcal{M}$ consists of the symmetrized product of states $\phi^{(i_n)}$ from each copy $\mathcal{M}$ in the product theory $\mathcal{M}^N$ such that

$$\Phi = \text{Sym}(\otimes_{n=1}^N \phi^{(i_n)}), \tag{4.7}$$

where n labels different copies of $\mathcal{M}$ and i_n denotes the state in the nth copy. For example, when $N = 2$ we have states of the form

$$\Phi = \Big\{\phi^{(i_1)} \otimes \phi^{(i_2)}, \frac{1}{2}\big(\phi^{(i_1)} \otimes \phi^{(j_2)} + \phi^{(j_1)} \otimes \phi^{(i_1)}\big)\Big\}, \tag{4.8}$$

where $i \neq j$. As a result of the symmetrization procedure, there is a vast reduction on the number of states in $\text{Sym}^N\mathcal{M}$ compared to the product theory $\mathcal{M}^N$. In order to illustrate this, consider the case where the seed CFT $\mathcal{M}$ has only m states $\phi^{(i_1)}$ with $i_1 = \{1, \dots, m\}$. Then the number of states in $\text{Sym}^N\mathcal{M}$ is given by

$$\sum_{k=1}^{m} \binom{m}{k}\binom{N-1}{k-1} = \binom{N+m-1}{m-1} \underset{N \gg m}{\sim} \frac{N^{m-1}}{(m-1)!}, \tag{4.9}$$

which grows polynomially in N when $N \gg m$. In contrast, in the product theory $\mathcal{M}^N$ the number of states grows exponentially in N as m^N.

Among the states in the untwisted sector we can distinguish between *single particle* and *multiparticle* states. The former corresponds to states where all but one state is the vacuum, which we denote by $\mathbb{I}$. These states take the form

$$\Phi = \text{Sym}(\phi^{(1)} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}), \tag{4.10}$$

and have conformal weight

$$h_\Phi = h_{\phi^{(1)}}, \qquad \bar{h}_\Phi = \bar{h}_{\phi^{(1)}}. \tag{4.11}$$

We can think of the single-particle state (4.11) as the sum of the state $\phi^{(1_n)}$ from each copy of $\text{Sym}^N\mathcal{M}$. The stress tensor T is one example of a single-particle state that is given by the sum of the stress tensors $T^{(i)}$ from each copy of the symmetric orbifold

$$T = \text{Sym}(T^{(1)} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}) = \sum_{i=1}^{N} T^{(i)}, \tag{4.12}$$

and similarly for $\bar{T} = \sum_{i=1}^{N} \bar{T}^{(i)}$. Using the fact that the OPE of the stress tensor in the seed CFT $\mathcal{M}$ is given by

$$T^{(1)}(z)T^{(1)}(0) \sim \frac{c_0}{2z^4} + \frac{2T^{(1)}(z)}{z^2} + \frac{\partial T^{(1)}(z)}{z}, \tag{4.13}$$

it's not difficult to verify that the OPE of the stress tensor of $\mathrm{Sym}^N\mathcal{M}$ is given by

$$T(z)T(0) = \sum_{i=1}^{N} T^{(i)}(z)T^{(i)}(0) \sim \frac{Nc_0}{2z^4} + \frac{2T(z)}{z^2} + \frac{\partial T(z)}{z}. \tag{4.14}$$

where we have used the fact that $T^{(i)}(z)T^{(j)}(0) = 0$ when $i \neq j$, i.e. the OPE between fields in different copies of $\mathrm{Sym}^N\mathcal{M}$ don't interact with each other. It follows that the central charge of the symmetric orbifold is indeed linear in N and givne by $c = Nc_0$.

Multiparticle states, on the other hand, are states made of the product of two or more states from different copies of $\mathrm{Sym}^N\mathcal{M}$ and take the general form (4.7). In particular, their conformal weights are given by the sum of the conformal weights from each copy, that is

$$h_\Phi = \sum_{j=1}^{n} h_{\phi^{(i_j)}}, \qquad \bar{h}_\Phi = \sum_{j=1}^{n} \bar{h}_{\phi^{(i_j)}}. \tag{4.15}$$

For example, a two-particle state made of the components of the stress tensor is

$$\mathrm{Sym}(T^{(1)} \otimes T^{(2)} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}) = \sum_{i,j}^{N} T^{(i)}T^{(j)}. \tag{4.16}$$

Exercise 4.2: one reason that symmetric orbifolds cannot be dual to semiclassical theories of gravity in three-dimensions is that, in the large-N limit, they contain an infinite number of higher spin currents. Use the OPE of the stress tensor to show that $W_4(z) \coloneqq \sum_{i=1}^{N}[(T^{(i)}T^{(i)})(z) - \frac{3}{10}\partial_z^2 T^{(i)}(z)] - \alpha \sum_{i\neq j}^{N} T^{(i)}(z)T^{(j)}(z)$ is a spin-4 current for an appropriate choice of α. Above $(AB)(z) \coloneqq \oint \frac{dw}{2\pi i} A(w)B(z)$ is the normal-ordered product of operators (see chapter 6 of Di Francesco et. al. for details). Find for which value of c_0 is $\alpha = 0$. This value corresponds to a well-known CFT with $N = 1$, which would suggest this model has a spin-4 current, in contradiction to what's found in the spectrum. Verify that for this value of c_0 the two-point function of W_4 vanishes, indicating that it corresponds to a null state that must be modded out of the spectrum.

We would now like to consider the contribution of single and multiparticle untwisted states to the partition function of $\mathrm{Sym}^N\mathcal{M}$. For generic states (4.7) where all of the $\phi^{(i_n)}$ are different, their contribution to the partition function can be obtained from $\frac{1}{N!}Z(\tau,\bar{\tau})^N$ where $Z(\tau,\bar{\tau})$ is the partition function of the seed CFT $\mathcal{M}$. This product of partition functions miscounts several states, however, since $Z(\tau,\bar{\tau})^N$ contains states that are symmetric under subgroups of S_N. The simplest example of these states are states of the form

$$\Phi = \otimes_{n=1}^{N} \phi^{(1)}, \tag{4.17}$$

where $\phi^{(1)}$ is some state in the seed CFT. Such states are invariant under any element of the symmetric group so dividing $Z(\tau,\bar\tau)^N$ by $N!$ not only miscounts them, but yields an unphysical noninteger number for these states. In order to account for this kind of states we can either subtract them from $Z(\tau,\bar\tau)^N$ before normalizing by $N!$ or add additional contributions to $\frac{1}{N!}Z(\tau,\bar\tau)^N$ that fixes their normalization.

In order to illustrate this and motivate the general formula, let's work out the example with $N=3$ in detail.[11] When $N=3$, we have the following kinds of untwisted states

(i) $\Phi_{(i)} \coloneqq \phi^{(i)} \otimes \phi^{(i)} \otimes \phi^{(i)}$
This kind of states consist of three copies of the same field $\phi^{(i)}$, one from each copy of $\mathrm{Sym}^3\mathcal{M}$. The simplest example of such a state is the vacuum, for which $\phi^{(i)} = \mathbb{I}$. The conformal weight of these type of states is

$$h^{(i)} = 3h_{\phi^{(i)}}, \qquad \bar h^{(i)} = 3\bar h_{\phi^{(i)}}. \tag{4.18}$$

Consequently, their contribution to the partition function can be obtained from the seed partition function $Z(\tau,\bar\tau)$ by multiplying the modular parameter by 3. In other words, the contribution of these states to the partition function, which we denote by $Z^{(3)}(\tau,\bar\tau)$, is given by

$$Z^{(3)}(\tau,\bar\tau) = Z(3\tau, 3\bar\tau). \tag{4.19}$$

(ii) $\Phi_{(i,j)} \coloneqq \mathrm{Sym}(\phi^{(i)} \otimes \phi^{(i)} \otimes \phi^{(j)})$ with $i \neq j$
These states consist of two copies of $\phi^{(i)}$ and one copy of $\phi^{(j)}$ such that the conformal weights are

$$h^{(i)} = 2h_{\phi^{(i)}} + h_{\phi^{(j)}}, \qquad \bar h^{(i)} = 2\bar h_{\phi^{(i)}} + \bar h_{\phi^{(j)}}. \tag{4.20}$$

Therefore, their contribution to the partition reads

$$Z^{(2)}(\tau,\bar\tau) \coloneqq Z(2\tau, 2\bar\tau;\mu) Z(\tau,\bar\tau) - Z^{(3)}(\tau,\bar\tau), \tag{4.21}$$

where $Z^{(3)}(\tau,\bar\tau)$ subtracts additional states with weights $(h^{(i)}, \bar h^{(i)}) = (3h_{\phi^{(i)}}, 3\bar h_{\phi^{(i)}})$ from $Z(2\tau,2\bar\tau)Z(\tau,\bar\tau)$.

(iii) $\Phi_{(i,j,k)} \coloneqq \mathrm{Sym}(\phi^{(i)} \otimes \phi^{(j)} \otimes \phi^{(k)})$ with $i \neq j \neq k$
These states consist of different fields from different copies of $\mathrm{Sym}^3\mathcal{M}$ such that their conformal weights are given by the general formulae (4.15). Their contribution to the

[11] The example $N=2$ is too simple and the same as the $\mathbb{Z}_2$ orbifold of $\mathcal{M}$.

partition function $Z^{(1)}(\tau,\bar\tau)$ is therefore given by

$$Z^{(1)}(\tau,\bar\tau) \coloneqq \tfrac{1}{3!}\big[Z(\tau,\bar\tau)^3 - 3Z^{(2)}(\tau,\bar\tau) - Z^{(3)}(\tau,\bar\tau)\big], \tag{4.22}$$

where we have once again subtracted the contributions of $\Phi_{(i,j)}$ and $\Phi_{(i)}$ from $Z(\tau,\bar\tau)^3$.

Altogether, the partition function in the untwisted sector of $\mathrm{Sym}^3\mathcal{M}$ reads

$$Z_3^{\text{untwisted}}(\tau,\bar\tau) = \sum_{i=1}^{3} Z^{(i)}(\tau,\bar\tau) = \frac{1}{3!}\Big[Z(\tau,\bar\tau)^3 + 3Z(2\tau,2\bar\tau)Z(\tau,\bar\tau) + 2Z(3\tau,3\bar\tau)\Big]. \tag{4.23}$$

In order to understand the meaning of this partition function we need to introduce some features of the symmetric group S_N. The symmetric group contains $N!$ elements that belong to one of $p(N)$ conjugacy classes. The conjugacy classes of S_N consist of the product of $\mathbb{Z}_{n_i}$ cycles with $n_i = 1,\dots,N$ such that $\sum_i n_i = N$.[12] As a result, the number of conjugacy classes $p(N)$ is the number of partitions of N whose generating functional is $\sum_{i=1}^{\infty} p(i)x^i = \prod_{j=1}^{\infty}(1-x^j)^{-1}$. Let $\{k_1,\dots,k_N\}$ label each of the conjugacy classes such that k_n counts the number of $\mathbb{Z}_n$ cycles in the class, i.e.

$$\{k_1,\dots,k_N\} \coloneqq \prod_{i=1}^{N} \mathbb{Z}_i^{k_i} = \mathbb{I}^{k_1}\cdots\mathbb{Z}_n^{k_n}, \qquad \sum_i k_i n_i = N. \tag{4.24}$$

Each conjugacy class $\{k_1,\dots,k_N\}$ has the following number of elements

$$\frac{N!}{\prod_{n=1}^{N} n^{k_n} k_n!}, \tag{4.25}$$

where the factors of n count the size of each $\mathbb{Z}_n$ cycle while $k_n!$ counts the permutations of the $\mathbb{Z}_n$ cycles. It's not difficult to show that the summing (4.25) over all conjugacy classes yields the rank of S_N, namely $N!$.

> **Exercise 4.3:** show that the sum of (4.25) over conjugacy classes of S_N is equal to $N!$, i.e. show that $\sum_{\{k_1,\dots,k_N\}} \frac{1}{\prod_{n=1}^{N} n^{k_n} k_n!} = 1$.

Let us now return to our S_3 example. In this case we have six elements that belong to one of three conjugacy classes such that

elements	size	conjugacy class	$\{k_1,k_2,k_3\}$
abc	1	$\mathbb{I}^3$	$\{3,0,0\}$
bac, cba, acb	3	$\mathbb{I}\cdot\mathbb{Z}_2$	$\{1,1,0\}$
bca, cab	2	$\mathbb{Z}_3$	$\{0,0,1\}$

(4.26)

[12] Here $\mathbb{Z}_1 = \mathbb{I}$ is identified with the trivial conjugacy class whose only element is the indentity.

The cycle index I_3 of S_3 packages all of this information into the following object

$$I_3 = \mathbb{I}^3 + 3\mathbb{I} \cdot \mathbb{Z}_2 + 2\mathbb{Z}_3. \tag{4.27}$$

We see that if we make the following identification

$$\mathbb{I} \leftrightarrow Z(\tau, \bar{\tau}), \qquad \mathbb{Z}_2 \leftrightarrow Z(2\tau, 2\bar{\tau}), \qquad \mathbb{Z}_3 \leftrightarrow Z(3\tau, 3\bar{\tau}), \tag{4.28}$$

then up to a normalization, the untwisted partition function (4.23) matches the cycle index of S_3 (4.27).

For general N, the untwisted sector torus partition function of $\mathrm{Sym}^N\mathcal{M}$ is proportional to the cycle index of S_N such that

$$Z_N^{\text{untwisted}}(\tau, \bar{\tau}) = \frac{I_N}{N!} = \frac{1}{N!} \sum_{\{k_1, \ldots, k_N\}} \frac{N!}{\prod_{n=1}^N n^{k_n} k_n!} \prod_{n=1}^{N} Z(n\tau, n\bar{\tau})^{k_n}, \tag{4.29}$$

where $\sum_i k_i n_i = N$ and we have used the following identification between partition functions and cycles of S_N

$$\mathbb{Z}_n \leftrightarrow Z(n\tau, n\bar{\tau}). \tag{4.30}$$

Note that this formula is universally valid for the untwisted sector of any $\mathrm{Sym}^N\mathcal{M}$ as it relies only on the structure of S_N and not on the details of the seed CFT. In particular, we can use the generating functional of the cycle index of S_N to write a generating functional for the untwisted partition function (4.29)

$$\mathcal{Z}^{\text{untwisted}}(\tau, \bar{\tau}; p) \coloneqq \sum_{N=0}^{\infty} p^N Z_N^{\text{untwisted}}(\tau, \bar{\tau}) = \exp\left(\sum_{n=1}^{\infty} \frac{p^n}{n} Z(n\tau, n\bar{\tau})\right). \tag{4.31}$$

Exercise 4.4: classify the untwisted states of $\mathrm{Sym}^4\mathcal{M}$, find their contribution to the torus partition function, and verify that it matches the cycle index of S_4 up to a normalization.

References

[1] A. Klemm and M. G. Schmidt, *Orbifolds by Cyclic Permutations of Tensor Product Conformal Field Theories*, *Phys. Lett. B* **245** (1990) 53.

[2] R. Dijkgraaf, G. W. Moore, E. P. Verlinde and H. L. Verlinde, *Elliptic genera of symmetric products and second quantized strings*, *Commun. Math. Phys.* **185** (1997) 197 [hep-th/9608096].

[3] P. Di Francesco, P. Mathieu and D. Senechal, *Conformal Field Theory*, Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997, 10.1007/978-1-4612-2256-9.

[4] R. Dijkgraaf, *Fields, strings, matrices and symmetric products*, hep-th/9912104.

[5] P. Bantay, *Symmetric products, permutation orbifolds and discrete torsion*, *Lett. Math. Phys.* **63** (2003) 209 [hep-th/0004025].