Lecture 8

In this lecture we continue with our study of symmetric product orbifolds. We begin by describing the twisted sector of these theories and its contribution to the torus partition function. We then show that these states are necessary to preserve modular invariance of the theory. One consequence of modular invariance is that the partition function is universal at large N without the necessity of imposing a sparseness condition on the light states. Finally, we consider the symmetric product orbifolds of $T\bar{T}$-deformed CFTs, construct their torus partition function, and obtain the spectrum of twisted states. We will see that modular invariance also implies a universal torus partition function of symmetric product orbifolds of $T\bar{T}$-deformed CFTs.

4.1.2 Twisted sector

The twisted sector of $\mathrm{Sym}^N\mathcal{M}$ consists of single and multi-particle states Φ_g made up of one or more twisted states $\sigma_{(n)}$ with $n \geq 2$ such that

$$\Phi_g = \mathrm{Sym}(\otimes_{i=1}^N \sigma_{(n_i)}), \qquad \sum_i n_i = N, \tag{4.32}$$

where $\sigma_{(1)}$ corresponds to an untwisted state from a copy of $\mathrm{Sym}^N\mathcal{M}$. Each twisted state $\sigma_{(n)}$ is identified with a $\mathbb{Z}_n$ cycle of S_N. As a result, every state Φ_g in the twisted sector corresponds to one of the conjugacy classes of S_N, i.e. it corresponds to a product of $\mathbb{Z}_{n_i}$ cycles. The twist field $\Phi_g(z)$ induces the following boundary conditions on the operators $\phi^{(i)}$ from each copy of the theory

$$\phi^{(j)}(e^{2\pi i}z)\Phi_g(0) = \phi^{(g(j))}\Phi_g(0), \qquad g \in S_N. \tag{4.33}$$

As a result, we can think of a twist field as gluing different copies of the theory. For example, the simplest twisted state in a $\mathrm{Sym}^N\mathcal{M}$ is

$$\Phi_{\mathbb{Z}_2} = \mathrm{Sym}(\phi_{(2)} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I}). \tag{4.34}$$

Inserting this operator glues two copies of $\mathrm{Sym}^N\mathcal{M}$ such that, when we take the double trace operator $\phi^{(1)}\otimes\phi^{(2)}\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I}$ around $z=0$, we end up with the operator $\phi^{(2)}\otimes\phi^{(1)}\otimes\mathbb{I}\otimes\cdots\otimes\mathbb{I}$.

For each field with conformal weights $(h, \bar{h})$ in the seed CFT $\mathcal{M}$, there is a twisted field $\sigma_{(n)}$ with conformal weights $(h_{(n)}, \bar{h}_{(n)})$ given by

$$h_{(n)} = \frac{h}{n} + \frac{c_0}{24}\left(n - \frac{1}{n}\right), \qquad \bar{h}_{(n)} = \frac{\bar{h}}{n} + \frac{c_0}{24}\left(n - \frac{1}{n}\right), \qquad h_{(n)} - \bar{h}_{(n)} \in n\mathbb{Z}. \tag{4.35}$$

One way to derive these formulae is by requiring modular invariance of the torus partition function, as we'll describe shortly. In order to gain an intuition for (4.35), consider instead the twisted energies obtained by adding the Casimir energy for the number of copies glued by the twist operator, that is

$$E_L^{(n)} = h_{(n)} - \frac{nc_0}{24} = \frac{E_L}{n}, \qquad E_R^{(n)} = \bar{h}_{(n)} - \frac{nc_0}{24} = \frac{E_R}{n}. \tag{4.36}$$

These energies reflect the fact that the twist operator is in some sense spread among n copies of $\mathrm{Sym}^N\mathcal{M}$, as illustrated in figure 11. We also note that insertions of the twist fields $\sigma_{(n)}$ in correlation functions of $\mathrm{Sym}^N\mathcal{M}$ can be dealt with by mapping the base space the theory is defined in to an n-sheeted branched covering via the covering map $z \to t_n$ where t is given by

$$t_n = z^{1/n} e^{2\pi i k/n}. \tag{4.37}$$

Under this map each copy of an untwisted field $\phi^{(i)}$ is mapped to a region of the cover space, while $\sigma_{(n)}$ is mapped to a field at the origin with conformal weights (4.35). This is illustrated in figure **??**.

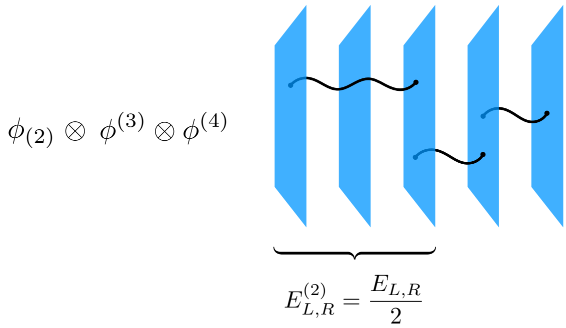

Figure 11: The energy of a twist-n state in a symmetric orbifold can be understood as the energy of state in the seed CFT $\mathcal{M}$ that is spread among n-copies of $\mathrm{Sym}^N\mathcal{M}$, as illustrated above for $n = 2$.

> **Exercise 4.5:** in theories with $\mathcal{N} \geq 2$ supersymmetry, an exactly marginal operator can be obtained from the supersymmetric descendant of a chiral primary with conformal weight $h = 1/2$. Use this fact to find the maximum allowed seed central charge for each value of the twist. Note that there is only a finite number of twists you need to consider since unitarity constraints the central charge to be $c_0 \geq 1$. Using this information, can you deduce the twist of the operator needed to deform the symmetric orbifold of T^4 discussed in the previous lecture?

Let us now consider the contribution of the twisted states to the torus partition function of $\mathrm{Sym}^N\mathcal{M}$. These states are needed to preserve the modular invariance of the partition function, the latter of which is required for consistency of the CFT. The partition function of the untwisted states (4.29) is not invariant under modular transformations because each of the $Z(n\tau, n\bar{\tau})$ terms with $n \geq 2$ is not modular invariant. While (4.29) is invariant under modular $\mathcal{T}$ transformations ($\tau \mapsto \tau + 1$), it fails to be invariant under $\mathcal{S}$ transformations for which $\tau \mapsto -1/\tau$ since

$$\mathcal{T} \cdot Z(n\tau, n\bar{\tau}) = Z(n\tau, n\bar{\tau}), \tag{4.38}$$

$$\mathcal{S}\cdot Z(n\tau,n\bar{\tau}) = Z\left(-\frac{n}{\tau},-\frac{n}{\bar{\tau}}\right) = Z\left(\frac{\tau}{n},\frac{\bar{\tau}}{n}\right) \tag{4.39}$$

where we used the modular invariance of the seed partition function $Z(\tau,\bar{\tau})$. In order to render the partition function invariant under modular transformations, we need to add additional terms to the partition function that correspond to the twisted states characteristic of orbifold theories.

Our strategy to make (4.29) modular invariant is to make each of the $Z(n\tau,n\bar{\tau})$ partition functions invariant under modular transformations. This can be accomplished by adding all of the modular images of $Z(n\tau,n\bar{\tau})$ that are not themselves equal to $Z(n\tau,n\bar{\tau})$. For convenience, let us assume that n is prime. Whereas $Z(n\tau,n\bar{\tau})$ is not invariant under $\mathcal{S}$ transformations, the following combination of modular images is $\mathcal{S}$-invariant

$$Z(n\tau,n\bar{\tau}) + Z\left(\frac{\tau}{n},\frac{\bar{\tau}}{n}\right). \tag{4.40}$$

This combination of partition functions is not invariant under $\mathcal{T}$ transformations however, since for any integers k and $\alpha\in[1,n-1]$ we have

$$\mathcal{T}^{\alpha+kn}\cdot\mathcal{S}\cdot Z(\tau,\bar{\tau}) = Z\left(\frac{\tau+\alpha}{n},\frac{\bar{\tau}+\alpha}{n}\right). \tag{4.41}$$

Therefore, the following combination of modular images is invariant under $\mathcal{T}$, $\mathcal{S}$, and $\mathcal{T}\cdot\mathcal{S}$ transformations

$$Z(n\tau,n\bar{\tau}) + \sum_{\alpha=0}^{n-1} Z\left(\frac{\tau+\alpha}{n},\frac{\bar{\tau}+\alpha}{n}\right). \tag{4.42}$$

It turns out that the combination of partition functions (4.42) is modular invariant under any combination of $\mathcal{T}$ and $\mathcal{S}$ transformations. In order to see this, let us consider an $\mathcal{S}$ transformation of (4.41)

$$\mathcal{S}\cdot\mathcal{T}^{\alpha+kn}\cdot\mathcal{S}\cdot Z(\tau,\bar{\tau}) = Z\left(\frac{\frac{\alpha\tau}{n}-\frac{1}{n}}{\tau},\frac{\frac{\alpha\bar{\tau}}{n}-\frac{1}{n}}{\bar{\tau}}\right) = Z\left(\frac{\alpha\tilde{\tau}-\frac{1}{n}}{n\tilde{\tau}},\frac{\alpha\bar{\tilde{\tau}}-\frac{1}{n}}{n\bar{\tilde{\tau}}}\right). \tag{4.43}$$

where we have defined $\tilde{\tau}=\tau/n$. It's convenient to perform several modular $\tilde{\mathcal{T}}$ transformations $\tilde{\tau}\mapsto\tilde{\tau}+1/n$ of (4.43) to put it into the following form[13]

$$\mathcal{S}\cdot\mathcal{T}^{\alpha+kn}\cdot\mathcal{S}\cdot Z(\tau,\bar{\tau}) = (\widetilde{\mathcal{T}})^{\frac{\tilde{\alpha}}{n}}\cdot Z\left(\frac{\alpha\tilde{\tau}-\frac{1}{n}(1+\alpha\tilde{\alpha})}{n\tilde{\tau}-\tilde{\alpha}},\frac{\alpha\bar{\tilde{\tau}}-\frac{1}{n}(1+\alpha\tilde{\alpha})}{n\bar{\tilde{\tau}}-\tilde{\alpha}}\right), \tag{4.44}$$

where $\tilde{\alpha}$ is a positive integer. The partition function on the RHS of (4.44) can be written as

$$Z\left(\frac{\alpha\tilde{\tau}-\frac{1}{n}(1+\alpha\tilde{\alpha})}{n\tilde{\tau}-\tilde{\alpha}},\frac{\alpha\bar{\tilde{\tau}}-\frac{1}{n}(1+\alpha\tilde{\alpha})}{n\bar{\tilde{\tau}}-\tilde{\alpha}}\right) = Z\left(\frac{a\tilde{\tau}+b}{c\tilde{\tau}+d},\frac{a\bar{\tilde{\tau}}+b}{c\bar{\tilde{\tau}}+d}\right), \tag{4.45}$$

[13]Note that a $\widetilde{\mathcal{T}}$ transformation is equivalent to the standard modular $\mathcal{T}$ transformation $\tau\mapsto\tau+1$.

where the a, b, c, and d parameters are given by

$$a = \alpha, \qquad b = -\frac{1}{n}(1 + \alpha\tilde{\alpha}), \qquad c = n, \qquad d = -\tilde{\alpha}, \qquad \alpha,\, \tilde{\alpha},\, n \in \mathbb{Z}. \tag{4.46}$$

These parameters satisfy $ad - bc = 1$ and would correspond to an $SL(2,\mathbb{Z})$ transformation provided that b is an integer. It turns out that when n is prime and $\alpha \in [1,\, n-1]$ is co-prime with n, it's always possible to find a pair of integers $(\tilde{\alpha},\, \tilde{k})$ such that

$$\alpha\tilde{\alpha} + 1 = \tilde{k}n. \tag{4.47}$$

This is known as Bezout's identity. Bezout's identity is telling us that b is an integer and that, given any $\alpha \in [1,\, n-1]$, there is a unique integer $\tilde{\alpha} \in [1,\, n-1]$ satisfying (4.47). The converse is also true, meaning that there is a one-to-one map between the integers α and $\tilde{\alpha}$ if both are restricted to the region $[1,\, n-1]$. Since the a, b, c, and d parameters describe a bona fide $SL(2,\mathbb{Z})$ transformation under which the seed partition function $Z(\tau,\bar{\tau})$ is modular invariant, we have

$$\mathcal{S} \cdot \mathcal{T}^{\alpha+kn} \cdot \mathcal{S} \cdot Z(\tau,\bar{\tau}) = (\widetilde{\mathcal{T}})^{\frac{\tilde{\alpha}}{n}} \cdot Z(\tilde{\tau},\bar{\tilde{\tau}}) = \mathcal{T}^{\tilde{\alpha}+kn} \cdot \mathcal{S} \cdot Z(\tau,\bar{\tau}) \tag{4.48}$$

where in the last equality we added a $\mathcal{T}^{kn}$ term since it leaves $\mathcal{S} \cdot Z(\tau,\bar{\tau})$ invariant. This is illustrated in figure **??**

We have shown that under a modular $\mathcal{S}$ transformation, the partition function function $\mathcal{T}^{\alpha+kn} \cdot \mathcal{S} \cdot Z(\tau,\bar{\tau})$ with $\alpha \in [1, n-1]$ transforms into $\mathcal{T}^{\tilde{\alpha}+kn} \cdot \mathcal{S} \cdot Z(\tau,\bar{\tau})$ with a generically different $\tilde{\alpha} \in [1, n-1]$. As a result, the linear combination of partition functions featured in (4.42) is modular invariant under any sequence of $\mathcal{T}$ and $\mathcal{S}$ transformations, and hence invariant under any modular transformation. The combination of modular images in (4.42) can be obtained by the action of the Hecke operator T'_n on the seed partition function $Z(\tau,\bar{\tau})$.[14] For an arbitrary integer n, the Hecke operator is defined by

$$(T'_n Z)(\tau,\bar{\tau}) = \sum_{\gamma|n} \sum_{\alpha=0}^{\gamma-1} Z\left(\frac{n\tau + \alpha\gamma}{\gamma^2}, \frac{n\bar{\tau} + \alpha\gamma}{\gamma^2}\right), \tag{4.49}$$

where $\gamma|n$ are the divisors of n. We see that when n is prime the action of the Hecke operator reduces to

$$(T'_n Z)(\tau,\bar{\tau}) = Z(n\tau, n\bar{\tau}) + \sum_{\alpha=0}^{n-1} Z\left(\frac{\tau+\alpha}{n}, \frac{\bar{\tau}+\alpha}{n}\right), \qquad n \in \mathbb{P}, \tag{4.50}$$

which is precisely the combination of modular images we found earlier in (4.42).

We will now prove that the Hecke transform of $Z(\tau,\bar{\tau})$ is modular invariant. The essence of

[14]We are using the convention of the Hecke operator usually found in the high energy physics literature, the latter of which differs from the standard one T_n by an overall rescaling, $T'_n = nT_n$.

the proof lies on the fact that the Hecke transform of $Z(\tau,\bar{\tau};\mu)$ can be equivalently written as

$$(T'_n Z)(\tau,\bar{\tau};\mu) = \sum_{A_1} Z(A_1\tau, A_1\bar{\tau}), \tag{4.51}$$

where we sum over a complete set of inequivalent elements of $\Gamma(n)$, which is the set of all transformations $\tau \mapsto M\tau = \frac{\rho\tau+\sigma}{\eta\tau+\delta}$ with $\rho\delta - \eta\sigma = n$ and $\rho,\sigma,\eta,\delta \in \mathbb{Z}$. The inequivalent elements of $\Gamma(n)$ can be parametrized by the upper triangular matrices

$$A_1 = \begin{pmatrix} \rho_1 & \sigma_1 \\ 0 & \delta_1 \end{pmatrix}, \qquad \rho_1\delta_1 = n, \quad \sigma_1 \in \mathbb{Z}\,(\text{mod } \delta_1). \tag{4.52}$$

Exercise 4.6: verify that the properties of ρ_1, σ_1, and δ_1 given in (4.52) imply that (4.51) is equivalent to (4.49) with $(\rho_1,\sigma_1,\delta_1) = (n/\gamma, \alpha, \gamma)$.

Theorem 6.9 of Apostol's book on Modular Functions states that, given $A_1 \in \Gamma(n)$ and a standard modular transformation represented by the matrix $M_1 \in \Gamma(1)$, it is always possible to find $M_2 \in \Gamma(1)$ and an upper triangular matrix $A_2 \in \Gamma(n)$ such that

$$A_1 M_1 = M_2 A_2, \tag{4.53}$$

where the map is one-to-one. These properties imply that, under a modular transformation $\tau \mapsto M_1\tau = \frac{a_1\tau+b_1}{c_1\tau+d_1}$, the Hecke-transformed partition function (4.51) satisfies

$$\begin{aligned}(T'_n Z)\left(\frac{a_1\tau+b_1}{c_1\tau+d_1}, \frac{a_1\bar{\tau}+b_1}{c_1\bar{\tau}+d_1}\right) &= \sum_{A_1} Z(A_1M_1\tau, A_1M_1\bar{\tau}) \\ &= \sum_{A_2} Z(M_2A_2\tau, M_2A_2\bar{\tau}) \\ &= \sum_{A_2} Z(A_2\tau, A_2\bar{\tau}),\end{aligned} \tag{4.54}$$

where we used (4.53) in the second line, and the modular invariance of $Z(\tau,\bar{\tau})$ in the third line. Note that since the map between A_1 and A_2 in (4.53) is one-to-one, the sum over A_1 in the first line of (4.54) can be written as a sum over A_2 in the second line, and both sums run over all inequivalent elements of $\Gamma(n)$. Consequently, the third line of (4.54) is nothing but $(T'_n Z)(\tau,\bar{\tau};\mu)$, so that the Hecke transform of the partition function (4.51) is modular invariant for any positive integer n, namely

$$(T'_n Z)\left(\frac{a\tau+b}{c\tau+d}, \frac{a\bar{\tau}+b}{c\bar{\tau}+d}\right) = (T'_n Z)(\tau,\bar{\tau}). \tag{4.55}$$

Let us come back to the partition function of $\text{Sym}^N\mathcal{M}$. We have seen that the contribution of the untwisted states is universal, i.e. valid for any symmetric orbifold, and given by (4.29). This partition function is not modular invariant because each of the $Z(n\tau, n\bar{\tau})$ terms in (4.29) with $n \geq 2$ is not invariant under modular $\mathcal{S}$ transformations. Nevertheless we have seen that there is a combination of modular images of $Z(n\tau, n\bar{\tau})$ obtained from the Hecke transform that is modular invariant. Consequently, the untwisted partition function (4.29) can be made modular invariant by replacing each of the $Z(n\tau, n\bar{\tau})$ by its modular invariant completion via the Hecke transform, namely

$$Z(n\tau, n\bar{\tau}) \mapsto (T'_n Z)(n\tau, n\bar{\tau}). \tag{4.56}$$

As a result, the modular invariant partition function $Z_N(\tau, \bar{\tau})$ of any symmetric product orbifold $\mathrm{Sym}^N \mathcal{M}$ is given by

$$Z_N(\tau, \bar{\tau}) = \sum_{\{k_1, \ldots, k_N\}} \frac{1}{\prod_{n=1}^N n^{k_n} k_n!} \prod_{n=1}^{N} (T'_n Z)(\tau, \bar{\tau})^{k_n}, \tag{4.57}$$

where $\sum_i k_i n_i = N$ and we have defined $(T'_1 Z)(\tau, \bar{\tau}) = Z(\tau, \bar{\tau})$ for convenience. We stress that (4.57) is universal and given by the cycle index of S_N provided that we identify each cycle $\mathbb{Z}_n$ of S_N with the nth Hecke transform of the seed partition function

$$\mathbb{Z}_n \leftrightarrow (T'_n Z)(\tau, \bar{\tau}). \tag{4.58}$$

This means, in particular, that we can use the generating functional of the cycle index to write a generating functional for the partition function (4.57),

$$\mathcal{Z}(\tau, \bar{\tau}) := \sum_{N=0}^{\infty} p^N Z_N(\tau, \bar{\tau}; \mu) = \exp\left(\sum_{n=1}^{\infty} \frac{p^n}{n} (T'_n Z)(\tau, \bar{\tau}) \right), \tag{4.59}$$

where, in analogy with the untwisted partition function of $\mathrm{Sym}^N \mathcal{M}$, we define $Z_0(\tau, \bar{\tau}) = 1$ and $Z_1(\tau, \bar{\tau}) = Z(\tau, \bar{\tau})$.

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