Jackiw-Teitelboim (JT) gravity

In 2d the Einstein-Hilbert action is topological,

$$\int d^2\sigma \sqrt{|g|}\, R = \chi \qquad (\chi \equiv \text{Euler characteristic})$$

In order to have non-trivial dynamics we need to add degrees of freedom, e.g. a dilaton ϕ. The action of flat space Jackiw-Teitelboim gravity we are interested in is described by

$$S_{JT} = \int_{\mathcal{I}} d^2\sigma \sqrt{|g|}\,(\phi R - \Lambda) - \Lambda R_0 \oint_{\partial\mathcal{I}} du \sqrt{g_{uu}}$$

- ϕ: dilaton
- Λ: vacuum energy
- ΛR_0: boundary counterterm, with R_0 a free parameter
- g_{uu}: induced metric at the boundary

The kinetic term is motivated by the fact that the dimensional reduction of the Einstein-Hilbert action in higher dimensions produces this term.

This action is closely related to the action of JT gravity used in near AdS_2

holography, which is given by

$$\tilde{S}_{JT} = \int_{\mathcal{I}} d^2x \sqrt{|g|}\left[\phi\left(R + \frac{2}{L^2}\right) + \Lambda\right] + 2\phi_b \oint_{\partial\Sigma} du \sqrt{g_{uu}}\, K + 2C_0 \oint_{\partial\Sigma} du \sqrt{g_{uu}}$$

$\Lambda = 0$

scale of AdS_2

boundary value of the dilaton

trace of the extrinsic curvature

The two actions are related by $\phi \to \phi + \phi_0$, which shifts Λ and ϕ_b by:

$$\Lambda \to \Lambda + \frac{2\phi_0}{L^2}, \qquad \phi_b \to \phi_b + \phi_0$$

Hence, the physical combination of parameters is

$$\bar{\Lambda} = \Lambda - \frac{2}{L^2}\phi_b$$

In near AdS_2 holography one sets $\Lambda = 0$. For the flat space limit $L \to \infty$ of JT gravity we are interested in we must choose $\phi_b = 0$ instead. Note that after taking the $L \to \infty$ limit it's no longer possible to shift Λ.

Let us now consider (flat space) JT gravity coupled to matter fields Ψ_i

$$S = S_{JT}[\phi, g] + \underbrace{S_{matter}[\Psi_i, g]}_{\text{no coupling to } \phi}$$

The EOM read

$$g_{\mu\nu}: \quad (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)\phi - \frac{\Lambda}{2} g_{\mu\nu} = \frac{1}{2} T_{\mu\nu} \qquad \rightarrow \quad T_{\mu\nu} \equiv -\frac{2}{\sqrt{|g|}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}$$

$$\phi: \quad R = 0$$

We see that the EOM of the dilaton sets all metric solutions to be flat. However the full gravitational dynamics of the theory also includes the dilaton which is non trivial.

Our goal will be to derive the contribution of gravity to the S-matrix of matter scattering on the Minkowski vacuum. The latter can be obtained by setting $T_{\mu\nu} = 0$

and is given in lightcone coordinates $\sigma^{\pm} \equiv \frac{\sigma^0 \pm \sigma^1}{\sqrt{2}}$ by

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \phi = -\frac{\Lambda}{4}\eta_{\mu\nu}\sigma^\mu\sigma^\nu + C \qquad \text{(integration constant)}$$

$$\phi = -\frac{\Lambda}{2}\sigma^+\sigma^- + C$$

(asymptotically nontrivial)

Note that the vacuum is not invariant under translations, so it is not clear how it is possible to obtain a flat space (Poincaré-invariant) S-matrix via JT gravity. This puzzle is resolved by taking into account the additional symmetries of the theory. In order to see this it's convenient to use the conformal gauge $g_{\mu\nu} = e^{\Omega}\eta_{\mu\nu}$.

The JT action is then given by

$$S_{JT} = \int d\sigma^+ d\sigma^- \left(4\phi\,\partial_+\partial_-\Omega - \Lambda e^{2\Omega} \right) + \dots$$

which is invariant under $\phi \to \phi + f_+(\sigma^+) + f_-(\sigma^-)$.

Thus, the vacuum is invariant under the combinated transformations

$$\sigma^{\pm} \to \sigma^{\pm} + a^{\pm}, \qquad \phi \to \phi + \frac{\Lambda}{2} a^{-} \sigma^{+} + \frac{\Lambda}{2} a^{+} \sigma^{-}.$$

The effect of JT gravity on the scattering problem is to provide a dynamical set of coordinates such that $\sigma^{\pm} \to X^{\pm}(\sigma^{\pm})$. This is similar to what happens for the critical string in the Polyakov formalism:

$$\underbrace{X^0, \; X^1}, \; X^i \quad \text{where} \quad i = 2, \dots 25$$

think of these as dynamical coordinates for the string (which in the static gauge become $(X^0, X^1) = (\sigma^0, \sigma^1)$).

(in the static gauge the string transforms non linearly under Poincaré)

In the conformal gauge the EOM become

$$\partial_+^2 \phi = -\frac{1}{2} T_{++}, \qquad \partial_-^2 \phi = -\frac{1}{2} T_{--}$$

$$\partial_+\partial_-\phi = \frac{1}{2}\left(\Lambda e^{\Omega} + T_{+-}\right) \qquad \partial_+\partial_-\Omega = 0.$$

Requiring all solutions to reduce to the vacuum at infinity implies that $\Omega = 0$ everywhere (i.e. all of the dynamics are in the dilaton).

We can then introduce the dynamical coordinates

$$\underbrace{X^{\pm} = \frac{2}{\Lambda}\partial_{\mp}\phi}_{X^{\pm} \to X^{\pm} + a^{\pm} \text{ under Poincare'}} \equiv \sigma^{\pm} + Y^{\pm} \longrightarrow \text{note } Y^{\pm} \to 0 \text{ at the boundary}$$

In terms of these variables the EOM become:

$$\partial_+ Y^- = -\frac{T_{++}}{\Lambda}, \qquad \partial_- Y^+ = -\frac{T_{--}}{\Lambda}, \qquad \partial_+ Y^+ = \partial_- Y^- = \frac{T_{+-}}{\Lambda}$$

In particular we see that $\sigma^{\pm}$ and $Y^{\pm}$ are related by a nonlocal change of coordinates. Integrating the first two equations

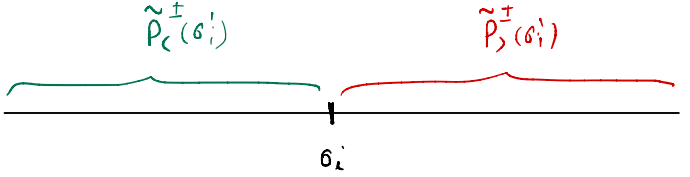
$$Y^{\pm}(\sigma^{\mp} = \infty) - Y^{\pm}(\sigma^{\mp} = -\infty) = \mp\frac{1}{2\pi}P^{\pm},$$

where $P^{\pm} \equiv \int_{-\infty}^{\infty} d\sigma^{\mp}\, T_{\mp\mp}$ is the total momentum along $\sigma^{\pm}$. We can then choose the integration constant such that

$$Y^{+}(\sigma^{0}=-\infty, \sigma_i') = \frac{1}{2\pi}\left[\tilde{P}^{+}_{>}(\sigma_i') - \tilde{P}^{+}_{<}(\sigma_i')\right]$$

$$Y^{-}(\sigma^{0}=-\infty, \sigma_i') = \frac{1}{2\pi}\left[\tilde{P}^{-}_{<}(\sigma_i') - \tilde{P}^{-}_{>}(\sigma_i')\right]$$

where $\tilde{P}^{\pm}_{>}(\sigma_i)$ denotes the momenta of all the particles with $\sigma' > \sigma_i'$ while $\tilde{P}^{\pm}_{<}(\sigma_i)$ are the momenta of particles with $\sigma' < \sigma_i'$ ($\beta < \beta_i$). equivalent to $\beta > \beta_i$



$p_i^0 = m\cosh\beta_i$

$p_i' = m\sinh\beta_i$

Note: the choice of integration constants leading to $Y^{\pm}(\sigma^{0}=-\infty, \sigma')$ above does not seem to be unique and the dressed S-matrix obtained below depends critically on this choice.

The dressed S-matrix

In the previous lecture, we coupled a Poincaré-invariant QFT to flat Jackiw-Teitelboim gravity

$$S = \int_{\mathcal{I}} d^2\sigma \sqrt{|g|}\,(\phi R - \Lambda) + S_{\text{matter}}[g, \psi_i]$$

The effect of JT gravity on the scattering problem at the nonperturbative level is to make the coordinates dynamical,

$$\sigma^{\pm} \quad \to \quad x^{\pm} \equiv \frac{2}{\Lambda}\partial_{\mp}\phi = \sigma^{\pm} + \gamma^{\pm}$$

Using the EOM, the $\gamma^{\pm}$ coordinates can be shown to satisfy

$$\gamma^{\pm}(\sigma^0 = -\infty, \sigma^1_i) = \pm\frac{1}{2\Lambda}\left[\tilde{P}^{\pm}_{>}(\sigma^1_i) - \tilde{P}^{\pm}_{<}(\sigma^1_i)\right]$$

where $\tilde{P}^{\pm}_{>}(\sigma_i)$ denotes the momenta of all the particles with $\sigma^1 > \sigma^1_i$ while $\tilde{P}^{\pm}_{<}(\sigma_i)$ are the momenta of particles with $\sigma^1 < \sigma^1_i$.

In the absence of gravity, an in state is defined (up to normalization) by

$$|p\rangle_{in} = a_i^\dagger(p)\,|0\rangle$$

where the creation operator $a_i^\dagger(p)$ is defined at $\sigma^0 \to -\infty$ via

$$\psi_i = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \frac{1}{\sqrt{2\epsilon}} \left[a_i^\dagger(p)\, e^{i p_\mu \sigma^\mu} + h.c. \right]$$

Once we turn on gravity and swicth to the dynamical coordinates we have

$$\psi_i = \int_{-\infty}^{\infty} \frac{dp}{\sqrt{2\pi}} \frac{1}{\sqrt{2\epsilon}} \left[\underbrace{a_i^\dagger(p)\, e^{i p_\mu Y^\mu}}_{\equiv A_i^\dagger(p)}\, e^{i p_\mu \sigma^\mu} + h.c. \right]$$

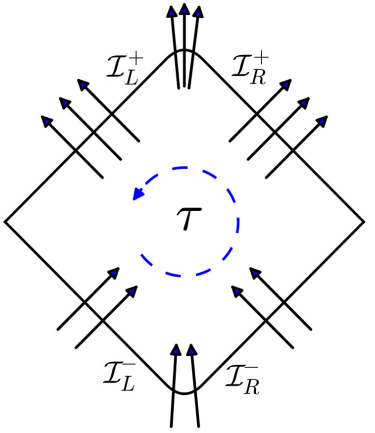
where $A_i^\dagger(p) \equiv a_i^\dagger(p)\, e^{i p_\mu Y^\mu} = a_i^\dagger(p)\, e^{-i(p^+ Y^- + p^- Y^+)}$ is the gravitationally-dressed creation operator.

Thus, for a general in state we have:

$$\widetilde{|\{p_i\}\rangle}_{in} = \prod_i A^\dagger(p_i)|0\rangle = e^{-\frac{i}{2\hbar}\sum_{i<j} p_i * p_j} |\{p_i\}\rangle_{in}, \qquad p_i * p_j \equiv \epsilon_{\mu\nu} p_i^\mu p_j^\nu$$

dressed in-state

ordered according to the rapidities β_i → compatible with the previous assumption



Similarly for the out states we have

$$\widetilde{|\{q_i\}\rangle}_{out} = \prod_i A^\dagger(q_i)|0\rangle = e^{\frac{i}{2\hbar}\sum_{i<j} q_i * q_j}$$

Altogether the dressed S-matrix is given by

$$\tilde{S} = {}_{out}\widetilde{\langle \{q_i\} | \{p_i\} \rangle}_{in} = {}_{out}\langle \{q_i\} | \{p_i\} \rangle \, e^{-\frac{i}{2\Lambda} \sum_{i<j} p_i * p_j - \frac{i}{2\Lambda} \sum_{i<j} q_i * q_j}$$

$$\tilde{S} = S \, e^{-\frac{i}{2\Lambda} \sum_{i<j} p_i * p_j - \frac{i}{2\Lambda} \sum_{i<j} q_i * q_j}$$

S: original S-matrix

$e^{\cdots}$: generalization of the CDD factor

Comments:

- The dressed S-matrix satisfies the physical conditions described in previous lectures, namely it's unitary, analytic, and crossing symmetric (provided the bare S-matrix also satisfies these properties).
- $\tilde{S}$ is the S-matrix of general $T\bar{T}$-deformed QFTs provided we identify

$$\mu = 2\Lambda^{-1}$$

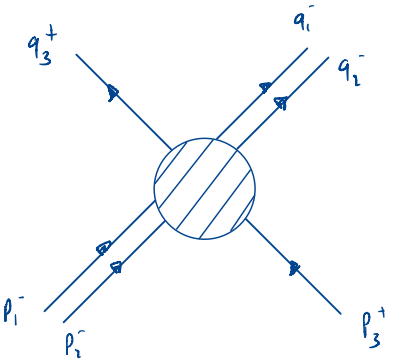
As a consistency check let us consider the scattering of 3 particles when the undeformed theory is a free scalar.

The gravitational dressing factor for the in states reads

$$U = \exp\Big\{ -\frac{i}{2\Lambda}\big(p_1^+ Y_1^- + p_1^- Y_1^+ + p_2^+ Y_2^- + p_2^- Y_2^+ + p_3^+ Y_3^- + p_3^- Y_3^+ \big)\Big\}$$

$$Y_1^- = -p_2^- - p_3^- \qquad Y_1^+ = p_2^+ + p_3^+$$

$$= \exp\Big\{ \frac{i}{\Lambda}\underbrace{\big(- p_1^- p_2^+ + p_1^+ p_2^- - p_1^- p_3^+ + p_1^+ p_3^- - p_2^- p_3^+ + p_2^+ p_3^+ \big)}_{\sum_{i<j} p_i * p_j}\Big\}$$



The momenta satisfy $p_1^- = q_1^-$, $p_2^- = q_2^-$, $p_3^+ = q_3^+$, so we can write

$$U(p_i) = e^{\frac{i}{4\Lambda}(s_{12}+s_{13}+s_{23})} \qquad \text{where} \quad s_{ij} = -(p_i+p_j)^2$$

$$U(q_i) = U(p_i)$$

Finally letting $\mu = 2\Lambda^{-1}$ we recover the S-matrix of a $T\bar{T}$-deformed free scalar

$$\tilde{S} = U(q_i)\, S\, U(p_i) = \mathbb{1}\, e^{i\mu(s_{12}+s_{13}+s_{23})/4}$$

Perturbative deformation

The derivation of the S-matrix above is non perturbative and holds to all orders in Λ^{-1}.

Perturbatively we can also show that coupling a QFT to flat JT gravity is equivalent to adding the $T\bar{T}$ operator to the classical action.

In order to see this we work with the original (non dynamical) coordinates $\sigma^{\pm}$ - the analog of the string in the static gauge - and expand the fields to linear order around the vacuum

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \qquad \phi = -\frac{\Lambda}{2}\sigma^{+}\sigma^{-} + c + \varphi$$

The quadratic action is then given by

$$S_{JT}^{(2)} = \int \varphi\left(\partial_{+}^{2} h_{--} + \partial_{-}^{2} h_{++} - 2\partial_{+}\partial_{-} h_{+-}\right) + \frac{\Lambda}{4}\left(h_{++}h_{--} - 2h_{+-}^{2}\right) + \frac{1}{2}h_{++}T_{--} + \frac{1}{2}h_{--}T_{++} + h_{+-}T_{+-}$$

$$+ \frac{\Lambda}{4}\left[\sigma^{+}h_{++}\left(2\partial_{-}h_{+-} - \partial_{+}h_{--}\right) + \sigma^{-}h_{--}\left(2\partial_{+}h_{+-} - \partial_{-}h_{++}\right)\right]$$

and the EOM read

$$\partial_+^2 h_{--} + \partial_-^2 h_{++} - 2\partial_+\partial_- h_{+-} = 0$$

$$\partial_\pm^2 \varphi + \frac{\Lambda}{4}\left[\left(2 \pm \sigma^+\partial_+ \mp \sigma^-\partial_-\right) h_{\pm\pm} + 2\sigma^\mp \partial_\pm h_{+-}\right] = -\frac{1}{2} T_{\pm\pm}$$

$$\partial_+\partial_- \varphi + \frac{\Lambda}{4}\left(2h_{+-} + \sigma^+\partial_- h_{++} - \sigma^-\partial_+ h_{--}\right) = 0$$

Using conservation of the stress tensor it is not difficult to show that the first equation is solved by

$$h_{\mu\nu} = -\frac{2}{\Lambda}\left(T_{\mu\nu} - \eta_{\mu\nu} T^\alpha{}_\alpha\right)$$

whereupon the other equations become

$$\partial_\pm^2 \varphi = \frac{1}{2}\left(1 + \sigma^\alpha \partial_\alpha\right) T_{\pm\pm}, \qquad \partial_+\partial_- \varphi = -\frac{1}{2}\left(1 + \sigma^\alpha \partial_\alpha\right) T_{+-}$$

$$\Downarrow$$

$$\partial_\pm \varphi = \frac{1}{2}\left(\sigma^\pm T_{\pm\pm} - \sigma^\mp T_{+-}\right)$$

Finally, plugging this solution back into the quadratic action we obtain

$$S_{J\bar{T}}^{(2)} = -\frac{1}{2\pi}\int d^2\sigma \left(T_{\mu\nu}T^{\mu\nu} - T_{\mu}^{\mu\,2}\right) = -\mu\int d^2\sigma \left(T_{\mu\nu}T^{\mu\nu} - T_{\mu}^{\mu\,2}\right) = S_{T\bar{T}}$$

which is the leading term in the $T\bar{T}$ deformation of the QFT coupled to $J\bar{T}$ gravity.