Aspects of holography and irrelevant deformations

Lecture 11:

$T\bar{T}$ and Jackiw - Teitelboim gravity III

References:

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The spectrum

We have argued that deforming a Poincaré-invariant QFT by the $T\bar{T}$ operator is equivalent to coupling the theory to flat Jackiw-Teitelboim gravity.

$$S = \int_{\Sigma} d^2\sigma \sqrt{|g|} \left(\phi R - \Lambda \right) + S_{matter}[g, \psi_i]$$

with the holographic identification

$$\Lambda = \frac{2}{\mu}.$$

Evidence for the correspondence comes from

- nonperturbative derivation of the dressed S-matrix; and
- reproducing the $T\bar{T}$ operator perturbatively to leading order in μ.

Modulo some subtleties, we will now show that the spectrum of a CFT coupled to flat JT gravity also reproduces the $T\bar{T}$ spectrum.

Let us consider flat JT gravity in the vielbein formalism where

$$g_{\mu\nu} = e_\mu^a \, e_\nu^b \, \eta_{ab} \quad , \qquad \omega_\mu{}^a{}_b = \omega_\mu \, \epsilon^a{}_b$$

vielbein

flat metric

spin connection

(top form in 2d)

In these variables, the action can be written (in Euclidean signature) as

$$S_{JT} = \int d^2\sigma \sqrt{|g|} \left(\phi R + \Lambda \right)$$

$$S_{JT} = \int d^2\sigma \left\{ \phi \, \epsilon^{\alpha\beta} \partial_\alpha \omega_\beta + \frac{\Lambda}{2} \epsilon^{\alpha\beta} \epsilon_{ab} \, e^a_\alpha e^b_\beta + \underbrace{\lambda^a \epsilon^{\alpha\beta} \left(\partial_\alpha e_{a\beta} - \epsilon_a{}^b \omega_\alpha e_{b\beta} \right)}_{\text{torsion - free constraint } T^a_{\alpha\beta} = 0} \right\}$$

The EOM read:

$$\lambda^a : \quad T^a{}_{\alpha\beta} = 0$$

$$\phi : \quad \epsilon^{\alpha\beta} \partial_\alpha \omega_\beta = 0 \qquad \Rightarrow \quad \omega_\alpha = \partial_\alpha \omega \quad (\text{flat})$$

$$\omega : \quad \partial_\alpha \phi = - \epsilon_{ab} \lambda^a e^b_\alpha$$

We can use the last equation to integrate out the dilaton ϕ.

Furthermore, we note that the vielbein formulation of gravitational theories comes with additional degrees of freedom, but also with additional gauge symmetries: local Lorentz transformations of the vielbeins

$$e^a_\alpha \to \Lambda^a{}_b\, e^b_\alpha, \qquad g_{\alpha\beta} = e^a_\alpha e^b_\beta \eta_{ab} \text{ is invariant}$$

In $d=2$ we have $\frac{d(d-1)}{2} = 1$ Lorentz transformations $\Lambda^a{}_b = \Lambda\, \epsilon^a{}_b$. We gauge fix this symmetry by choosing

$$\omega = 0 \qquad \Rightarrow \qquad \omega_\alpha = 0.$$

Altogether, the JT action reduces to

$$S_{JT} = \int d^2\sigma\, \epsilon^{\alpha\beta} \left(\lambda^a \partial_\alpha e^a_\beta + \frac{\Lambda}{2} \epsilon_{ab}\, e^a_\alpha e^b_\beta \right).$$

In contrast to the original vielbein action, we see that the action has a pair of shift symmetries:

$$\lambda^a \to \lambda^a + c^a \qquad \text{where } c^a \text{ are constants}$$

In analogy with the derivation of the S-matrix, this suggests the following identification of dynamical (or target space) coordinates:

$$\tilde{X}^a = \Lambda^{-1} \epsilon^a{}_b \lambda^b$$

Indeed, in terms of these variables, the EOM become

$$\partial_+ \tilde{X}^- = -\frac{T_{++}}{\Lambda}, \qquad \partial_- \tilde{X}^+ = -\frac{T_{--}}{\Lambda}, \qquad \partial_+(\tilde{X}^+ - \sigma^+) = \partial_-(\tilde{X}^- - \sigma^-) = \frac{T_{+-}}{\Lambda}$$

where we used $e^{\pm}_{\pm} = e^{\alpha \pm \beta} = 1$ together with

conformal gauge

$$\frac{\epsilon^{\alpha\beta}}{\sqrt{|g|}} \partial_\alpha \lambda^a = T^{\alpha\beta} e^a_\alpha .$$

More generally:

$$\partial_\alpha X^a = e^a_\alpha + \frac{2}{\Lambda} \epsilon^{ab} \epsilon_{\alpha\beta} T^{\beta\gamma} e_{b\gamma}$$

These are the same EOM we found in the metric formalism for the following choice of dynamical coordinates

$$X^{\pm} = 2\Lambda^{-1} \partial_{\mp} \phi .$$

Hence, the $\tilde{X}^a$ and X^μ coordinates are equivalent on-shell. In particular, the derivation of the dressed S-matrix is the same using the $\tilde{X}^a$ or X^μ coordinates.

The dynamical coordinates X^μ are defined only in the conformal gauge which makes the analysis of the torus partition function more complicated (the latter is used in the derivation of the spectrum).

On the other hand, $\tilde{X}^a$ are shift symmetric ($\tilde{X}^a \to \tilde{X}^a + c^a$) in any gauge and hence provide a gauge invariant definition of the dynamical coordinates.

Up to total derivative terms, the action of flat JT gravity coupled to matter reads

$$S = \frac{\Lambda}{2} \int d^2\sigma \; \epsilon^{\alpha\beta} \epsilon_{ab} \left(\partial_\alpha X^a - e_\alpha^{\;a} \right)\left(\partial_\beta X^b - e_\beta^{\;b} \right) + S_{\text{matter}}[g, \psi_i]$$

added total derivative term $\epsilon^{\alpha\beta} \epsilon_{ab} \partial_\alpha X^a \partial_\beta X^b$

The path integral definition of a $T\bar{T}$-deformed QFT is proposed to be given by

$$Z_{T\bar{T}}(\mu) = \int \frac{\mathcal{D}X^a \, \mathcal{D}e^a_\alpha \, \mathcal{D}\Psi_i}{V_{\text{Diff}}} e^{-S} = \int \frac{\mathcal{D}X^a \, \mathcal{D}e^a_\alpha}{V_{\text{Diff}}} e^{-\frac{1}{\mu}\int d^2\sigma \, \epsilon^{\alpha\beta}\epsilon_{ab}(\tilde{\partial}_\alpha X^a - e^a_\alpha)(\tilde{\partial}_\beta X^b - e^b_\beta)} Z_{\text{matter}}[g]$$

In order to derive the spectrum, we place the theory on the torus where

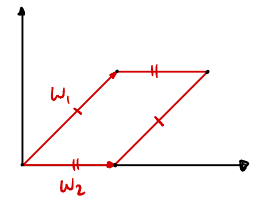
- we integrate over all vielbeins on a "worldsheet" with the topology of a torus;
- we integrate over all the dynamical coordinates mapping the wordsheet to a "target space" torus.

Let us sketch the steps necessary in the derivation of the spectrum. First we gauge fix the diffs so that the metric reads

$$\tilde{g}_{\alpha\beta} = e^{2\Lambda(\sigma)} \begin{pmatrix} 1 & \bar{\tau}_1 \\ \bar{\tau}_1 & \bar{\tau}_1^2 + \bar{\tau}_2^2 \end{pmatrix}$$

Weyl factor

Here we've put the modular parameter of the torus in the metric instead of in the identification of coordinates, $0 \leq \sigma^\alpha |n|^{1/2} \leq 1$.



$$\bar\tau = \bar\tau_1 + i\bar\tau_2 = \frac{w_2}{w_1}$$

As a result, the vielbein can be written as

$$\tilde e^a{}_\alpha = e^{\Omega(\sigma)} \underbrace{\left(e^{\epsilon\varphi(\sigma)}\right)^a{}_b}_{\text{local so(2) rotation}} \hat e^b{}_\alpha(\bar\tau), \qquad \hat e^a{}_\alpha = \begin{pmatrix} 1 & \bar\tau_1 \\ 0 & \bar\tau_2 \end{pmatrix}$$

Note that the so(2) factor is not a gauge symmetry (recall we fixed the gauge $w=0$) so that we must integrate over $\varphi(\sigma)$ in the path integral. The additional integration over $\varphi(\sigma)$ is what distinguishes the JT path integral between the metric and vielbein formalisms. In particular, integrating out $\varphi(\sigma)$ yields a path integral that depends only on metric variables but with a novel integration measure (not found in the metric formalism). As a result, the path integral in the vielbein formalism seems to be inequivalent to the one in the metric formalism.

We now evaluate the path integral by integrating over De and DX. The De integration becomes

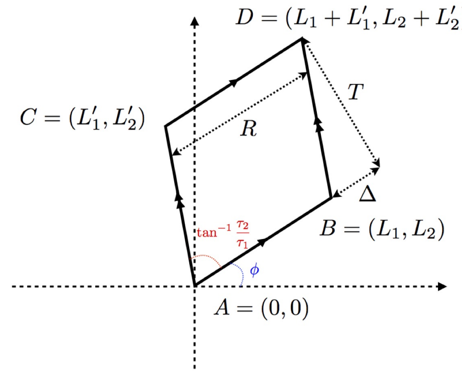
$$De \to D\Omega\, D\varphi\, d^2\bar{\tau}$$

The path integral turns out to localize on constant metrics and vielbeins such that $\Omega \to \bar{\Omega}$, $\varphi \to \bar{\varphi}$ where $\bar{\Omega}$ and $\bar{\varphi}$ are constant. The final result is

$$Z_{T\bar{T}}(\mu) = \frac{2}{|\mu|} \frac{t\, e^{-2t/\mu}}{(2\pi)^2} \int_{-\infty}^{\infty} d\bar{\Omega}\, e^{2\bar{\Omega}} \int_0^{2\pi} d\bar{\varphi} \int_{P} \frac{d^2\bar{\tau}}{\bar{\tau}_2} \exp\left[\frac{\mu}{|\mu|} \epsilon^{\alpha\beta} \epsilon_{ab} \left(\sqrt{\frac{2}{|\mu|}} L^a_\alpha \bar{e}^b_\beta - \frac{1}{2} \bar{e}^a_\alpha \bar{e}^b_\beta \right) \right] Z_{matter}$$

$\hookrightarrow$ upper-half plane : $-\infty < \bar{\tau}_1 < \infty$, $\bar{\tau}_2 > 0$.

where A is the area of the target space torus which is parametrized by $L^a_\alpha = (L_\alpha, L'_\alpha)$ such that $A = L_1 L_2' - L_1' L_2$ and



Since the integral above is over constant vielbeins (parametrized by $\bar\Omega$ and $\bar\Phi$) we can write:

$$\bar e^a_\alpha \equiv \sqrt{\frac{2}{|\mu|}}\, \bar L'^a_\alpha = \sqrt{\frac{2}{|\mu|}}\,(\bar L_\alpha, \bar L'_\alpha)$$

4 constants to replace $\bar\Omega, \bar\Phi, \bar\tau_1, \bar\tau_2$.

We then obtain

$$Z_{T\bar T}(\mu) = \frac{4}{\mu^2} A\, e^{-\frac{2A}{\mu}} \int_{\bar A > 0} \frac{d^4 \bar L}{(2\pi)^2 \bar A}\, e^{-\frac{2}{\mu}\epsilon^{\alpha\beta}\epsilon_{ab}(L^a_\alpha \bar L^b_\beta - \bar L^a_\alpha \bar L^b_\beta)}\, Z_{matter}[\bar g_{\alpha\beta}]$$

Note that the integral above is formally divergent since the exponential term does not have a definite sign. This is a "standard" problem of Euclidean quantum gravity.

The strategy to evaluate the integral is to use the saddle point approximation and hope for the best, namely, assume that one can take the right integration contours that regulate the integral and render it finite.

Let us change variables $(L, L') \to (L, \tau)$ and $(\bar{L}, \bar{L}') \to (\bar{L}, \bar{\tau})$ where τ is the standard modular parameter of the target space torus such that

$$Z_{T\bar{T}}(\mu) = \sum_n \frac{4A}{(2\pi)^2 \mu^4} \, e^{2A/\mu^2} \int_{-\infty}^{\infty} d^2\bar{L} \int_P \frac{d^2\bar{\tau}}{\bar{\tau}_2} \, e^{\frac{2}{\mu^2}\left\{ \bar{R}\bar{\tau}_2 - R\left[\bar{L}_1(\bar{\tau}_2 + \tau_2) + \bar{L}_2(\bar{\tau}_1 - \tau_1)\right]\right\}} \, e^{-\bar{\tau}_2 \bar{R} E_n(\bar{R}) + 2\pi i k_n \bar{\tau}_1}$$

where R is the size of the spatial circle $R = \sqrt{L_1^2 + L_2^2}$ and we used

$$Z_{matter}[g_{\alpha\beta}] = \sum_n e^{-\bar{\tau}_2 \bar{R} E_n(\bar{R}) + 2\pi i \bar{R} \bar{J}_n \bar{\tau}_1} , \qquad \bar{J}_n = \frac{k_n}{\bar{R}}$$

Let us assume the undeformed matter theory is a CFT, namely

$$E_n = \mathcal{E}_n R^{-1}.$$

Then the $\bar{\tau}$ integral is Gaussian and yields

$$Z_{T\bar{T}}(\mu) = \sum_n \frac{A}{2\pi\mu^2} e^{2A/\mu^2} \int_{\mathcal{P}} \frac{d^2\bar{\tau}}{\bar{\tau}_2^2} e^{-(R^2/2\mu^2\bar{\tau}_2)\left[(\bar{\tau}_1-\tau_1)^2+(\tau_2-\bar{\tau}_2)^2\right]} e^{-\bar{\tau}_2\mathcal{E}_n + 2\pi i k_n \bar{\tau}_1}$$

Finally, performing the integral over $\bar{\tau}$ we obtain

$$Z_{T\bar{T}}(\mu) = \sum_n \exp\left[2\pi i k_n \tau_1 - \underbrace{\frac{\tau_2 R^2}{\mu^2}\left(\sqrt{1+\frac{2\mu^2\mathcal{E}_n}{R^2}+\frac{4\pi^2\mu^4 k^2}{R^4}}-1\right)}_{\tau_2\mathcal{E}_n(\mu)}\right]$$

$\mathcal{E}_n(\mu) \equiv T\bar{T}$ spectrum

which is the partition function of a $T\bar{T}$-deformed CFT.

For general QFTs, the partition function can be evaluated by a saddle-point approximation and can be shown to yield, modulo assumptions, the desired result.

Together with the derivation of the S-matrix and the derivation of the infinitesimal deformation, we conclude that the $T\bar{T}$ deformation of a general QFT is equivalent to coupling the theory to flat Jackiw-Teitelboim gravity.