Aspects of holography and irrelevant deformations

Lecture 3:

massless S-matrices in 2d

References:

1703.1054, 1205.6805, Zamolodchikov '91

14.04.2021

The "simplest" toy model of quantum gravity is the theory whose spectrum at the quantum level is given by the lightcone spectrum

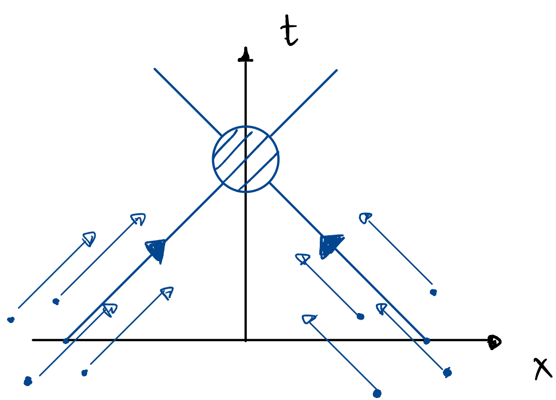
$$E_{LC} = \sqrt{\frac{4\pi^2 (N - \tilde{N})^2}{R^2} + \frac{R^2}{l_s^4} + \frac{4\pi}{l_s^2}\left(N + \tilde{N} - \frac{D-2}{12}\right)}$$

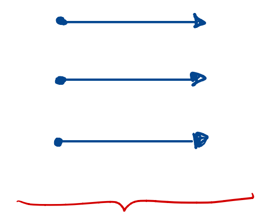
- When $D = 26$, this theory is the NG string in static gauge, where we assume $E_{NG} = E_{LC}$ holds beyond the classical level.
- Otherwise, this is a theory with $D-2$ dofs in the IR that lacks nonlinearly realized Poincaré, i.e. it's not the NG string.
- For any D, the theory with E_{LC} is a $T\bar{T}$-deformed CFT!

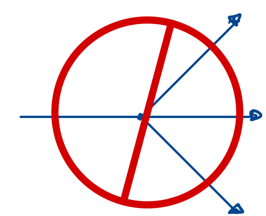
Last time we learned that the S-matrix for the scattering of massless particles in 2d exists provided that:

(1) left (right) movers sit on top of each other but don't inteact among themselves; and

(2) there is no particle production (no 1 → many amplitude).



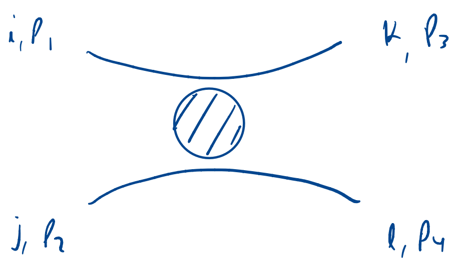




Both of these properties are satisfied by the NG string (the first to all orders while the second perturbatively).

Additionally, the $2 \to 2$ amplitude of the NG string in $D = 3, 26$ or the NG + PS string in other D, does not feature annihilations (to one loop).

All of this suggests that the scattering is purely elastic and $S = \mathbb{1}\, e^{2i\delta(s)}$.



$$\sim \delta(p_1 - p_3)\,\delta(p_2 - p_4)\; e^{2i\delta(s)}$$

This means that a string oscillating in one direction keeps oscillating forever.

Nonpertubatively, the lightcone spectrum implies:

(1) $a^{i\dagger}_n a^{j\dagger}_m \dots |0\rangle$ are eigenstates of the Hamiltonian

$\Rightarrow$ no particle production!

(2) $a^{i\dagger}_n a^{j\dagger}_m |0\rangle$ and $a^{k\dagger}_{n+m} |0\rangle$ are exactly degenerate (same N)

$\Rightarrow$ no annihititations!

$$N = \sum_{i=2}^{D-2} \sum_{n=1}^{\infty} n \, a^{i\dagger}_n a^i_n$$

These observations generalize the perturbative arguments for a diagonal S-matrix $S = \mathbb{1} \, e^{i 2\delta(s)}$.

We now take a look at the most general form of this S-matrix.

The S-matrix

The S-matrix is the unitary operator that takes an incoming state $|in\rangle = |i,j\rangle$ to the out state $|out\rangle = |k,l\rangle$, i.e. $|out\rangle = S\,|in\rangle$.

We can write the S-matrix in terms of the scattering amplitude $M_{ij,kl}$

$$S = \mathbb{1} + i\,\delta^{(2)}(k_1 + k_2 - k_3 - k_4)\, M_{ij,kl}$$

several general properties of the S-matrix follow from M.

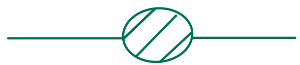
For example, from the Källen-Lehmann spectral representation of a typical field theory (see e.g. Peskin & Schroeder ch. 7)

$$\langle T\phi(x)\,\phi(y)\rangle = \int_0^\infty \frac{dM^2}{2\pi}\,\rho(M^2)\,D_F(x-y)$$

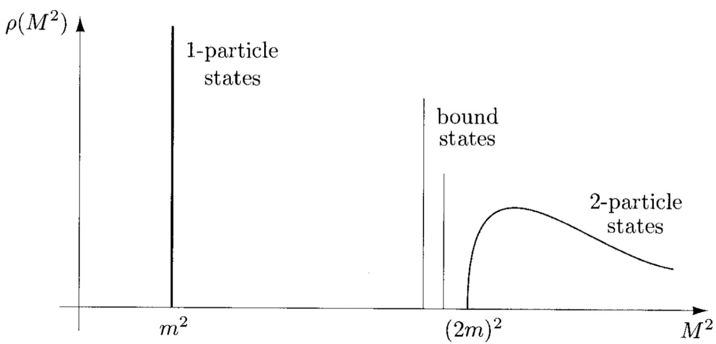
full propagator

spectral density

Feynman propagator

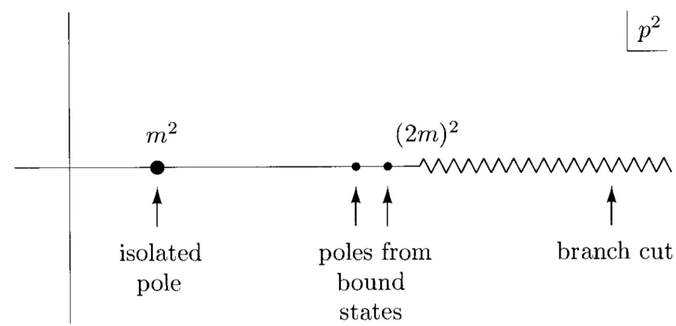


spectral density



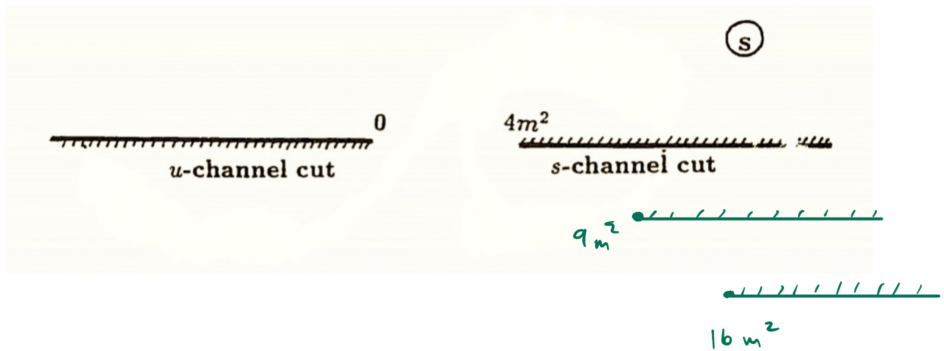
$$\rho(M^2) \sim \delta(M^2-m^2) + \dots$$

propagator (momentum space)

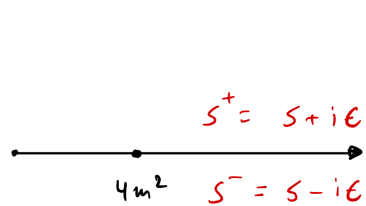


$$\frac{i}{p^2-m^2-i\epsilon} + i\int dM^2\,\frac{\rho(M^2)}{p^2-M^2-i\epsilon}$$

This leads to the following analytic structure of the S-matrix for the $2 \to 2$ scattering of massive particles of mass m (without particle production)



More generally, the analytic structure of the S-matrix is a consequence of unitarity and analiticity. Let us define $s^{\pm} = s \pm i\epsilon$ such that



where $s \geq 4m^2$

Then:

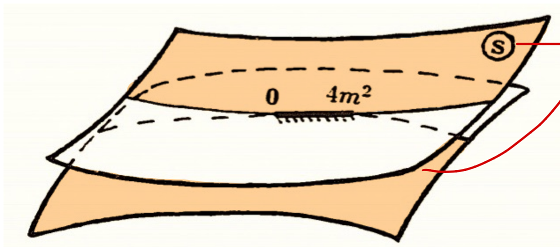
- unitarity implies that: $S(s^+)\, S^\dagger(s^+) = S^\dagger(s^+)\, S(s^+) = 1$.
- while $\underbrace{\text{analiticity}}_{}$ yields: $\underbrace{S(s^{+*}) = S(s^+)^\dagger}_{S(s^+)\,S(s^-) = 1}$ (Hermitean analiticity)

(locality: amplitudes are real boundary values of analytic functions of complex variables)

Consequently there is a branch cut of the square-root type from $s = 4m^2$ to ∞.

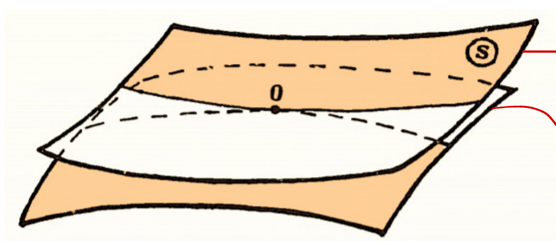
Using crossing symmetry $S(s) \sim S(u) = S(4m^2 - s)$, there is another branch cut from $s = 0$ to $-\infty$. Hence we obtain the analytic structure of the S-matrix.

For purely elastic scattering (no $9m^2$, $16m^2$, ... branch cuts) the s-plane has 2 sheets (square root branch cut) and we can replace the cuts such that



physical sheet

In the massless case these branch points merge



S-matrix on the first sheet : S_1

on the second sheet : S_2

In other words, the S-matrix on the physical sheet is parametrized as

$$S(s) = \begin{cases} S_1(s) & \text{if} \quad \mathrm{Im}(s) > 0 \\ S_2(s) & \text{if} \quad \mathrm{Im}(s) < 0 \end{cases}$$

Then unitarity, analiticity, and crossing symmetry imply

$$S_1(s)\, S_2(s) = 1, \quad S_1(s) = S_2(-s) \quad \Longrightarrow \quad S_1(s)\, S_1(-s) = S_2(s)\, S_2(-s) = 1$$

Zamolodchikov '91 :

$$S_1 = \mathbb{1}\, e^{i\delta(s)} = \mathbb{1} \prod_j \frac{\mu_j + s}{\mu_j - s}\, e^{i\,P(s)}, \qquad P(-s) = -P(s)$$

↳ μ_j on the lower half of the complex plane (no poles).

Locality: $P(s)=0$ such that the corresponding amplitude is exponentially bounded. Intuitively, $e^{i\,P(s)}$ corresponds to interactions of the form $e^{i\,P(\partial^2)}$, which seem nonlocal – they originate from an infinite number of derivatives in the action.

The simplest example of this is the theory of a Goldstino, a massless fermion with nonlinearly realized susy, for which $P(s)=0$ and

$$S = \mathbb{1}\,\frac{iM^2 - s}{iM^2 + s}\,, \qquad M = \text{scale of susy breaking}$$

For the NG string (or the theory with E_{LC}) what are the μ_j and $P(s)$?