Aspects of holography and irrelevant deformations

Lecture 4:

the S-matrix

References:

1703.1054, 1205.6805, Zamolodchikov '91

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Last time we learned that unitarity, analiticity, and crossing symmetry imply that the S-matrix can be written as (here $\mathrm{Im}\, s > 0$)

$$S = \mathbb{1}\, e^{i\delta(s)} = \mathbb{1} \prod_j \frac{\mu_j + s}{\mu_j - s}\, e^{i P(s)}, \qquad P(-s) = -P(s), \qquad \mathrm{Im}\, \mu_j < 0$$

The simplest example of such an S-matrix compatible with locality ($P(s) = 0$) comes from the Goldstino $S_G = \mathbb{1}\, \dfrac{iM^2 - s}{iM^2 + s}$.

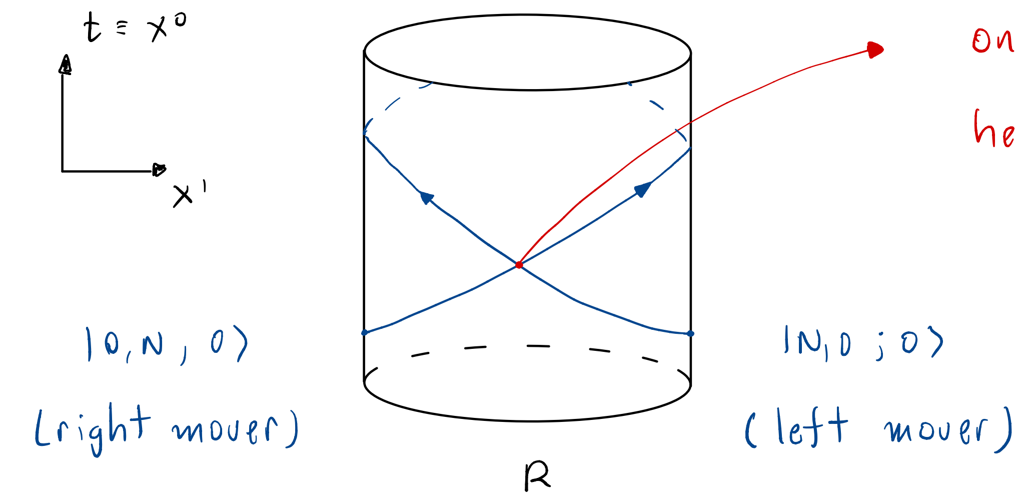
Today we will derive and study some of the implications of the simplest S-matrix with $P(s) \neq 0$, namely the S-matrix for the infinitely long free bosonic string ($D=26$) or the theory with the lightcone spectrum (any D).

Derivation of the S-matrix

Consider a 2-particle state $|N, N; 0\rangle$ $(\tilde{N} = N)$ and evolve it in time

$$|N, N; t\rangle = e^{-i\,\epsilon_{LC}(N,N)}\,|N, N; 0\rangle .$$

We can also think of this state as follows:



one interaction every $\Delta t = R/2$;

hence a phase shift

$$\exp\left(2 i \delta_{LC} \times \frac{2t}{R}\right)$$

If we let $R\to\infty$ $(t\to\infty)$ then we should obtain the same state:

$$|N,N;t\rangle = e^{-i[\Delta E_{LC}(N,0)+\Delta E_{LC}(0,N)+E_{LC}(0,0)-4\delta_{LC}/R]t}\,|N,N;0\rangle$$

energies of 1-particle states above the vacuum, e.g.

vacuum energy

$$\Delta E_{LC}(N,0) \equiv E_{LC}(N,0) - E_{LC}(0,0)$$

independent of constant shifts of E_{LC} (important later)

Comparing the two expressions we find

$$2\delta_{LC} = -\lim_{R\to\infty}\frac{R}{2}\Big[\underbrace{E_{LC}(N,N)-\Delta E_{LC}(N,0)-\Delta E_{LC}(0,N)-E_{LC}(0,0)}_{\text{binding energy of the 2-particle state}}\Big]$$

$$\boxed{2\delta_{LC} = \frac{5\,l_s^2}{4}}$$

The S-matrix of the critical NG string (or the theory with the lightcone spectrum E_{LC}) is given by:

$$S_{LC} = \mathbb{1}\, e^{2i\delta_{LC}} = \mathbb{1}\, e^{is\ell_s^2/4}$$

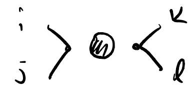
Hence $\mu_j = 0$ and $P(s) = s\ell_s^2/4$.

Comments:

1. Simplest example of a unitary, analytic, and crossing symmetric massless S-matrix in 2d with a nontrivial $P(s)$.

2. The S-matrix is defined all the way to the UV despite the theory being renormalizable.

3. Mildest possible violation of exponential boundedness. Note S is still bounded for $\mathrm{Im}(s) > 0$ (the physical sheet).

4. The S-matrix features no annihilations, meaning the corresponding amplitude $M_{ij,kl} = 0\,\delta_{ij}\,\delta_{kl} + \dots$ (i.e. no  contributions). The action corresponding to such S-matrix/spectrum is:

$$S_{LC} = S_{NG} + S_{PS} + \mathcal{O}(\ell_s^2)$$

Polchinski-Strominger term:

$$S_{PS} = -\frac{(D-26)}{192\pi}\int d^2\sigma \sqrt{-h}\, R\,\frac{1}{\Box}\,R$$

additional terms determined order by order

Perturbative check on the S-matrix

The $2 \to 2$ amplitude of the $D=26$ NG string on the "t-channel" $t=0$ reads:

$$\mathcal{M}_{ij,kl} = \cancel{A}^{\,0} \delta_{ij,kl} + B\, \delta_{ik,jl} + \cancel{C}^{\,0} \delta_{il,jk}$$

$$B = \frac{l_s^2}{2} s^2 + \frac{i\, l_s^4}{16} s^3 + \mathcal{O}(l_s^6)$$

Hence the S-matrix reads

$$S = \underbrace{\delta_{ik}\delta_{jl}\, \delta(p_1-p_3)\, \delta(p_2-p_4)}_{\mathbb{1}} + i\, \delta(p_1-p_3)\, \delta(p_2-p_4) \frac{\mathcal{M}_{ij,kl}}{2s}$$

$$= \mathbb{1} \left(1 + i \frac{s\, l_s^2}{4} - \frac{s^2 l_s^4}{32} + \dots \right)$$

$$= \mathbb{1}\, e^{i s l_s^2/4}$$

Evidence for a theory of quantum gravity

Before we extract some of the consequences of the S-matrix, we take a closer look at the spectrum.

The Goldstino

Theory of a single fermion that nonlinearly realizes spontaneously broken susy

$$\partial = \tfrac{1}{2}(\partial_\tau + \partial_\sigma)$$

$$\bar\partial = \tfrac{1}{2}(\partial_\tau - \partial_\sigma)$$

$$S_G = \frac{1}{2\pi}\int d^2\sigma \left\{ \psi\bar\partial\psi + \bar\psi\partial\bar\psi \underbrace{- \frac{1}{\pi^2 M^2}\overbrace{(\psi\partial\psi)(\bar\psi\bar\partial\bar\psi)}^{T\bar T} + \frac{\#}{M^4}(\psi\partial^3\psi)(\bar\psi\bar\partial^3\bar\psi) + \dots}_{\text{nonrenormalizable interactions}} \right\}$$

Using TBA we can extract the finite volume ($\sigma \sim \sigma + R$) spectrum

from the S-matrix $S = \mathbb{1}\,\dfrac{iM^2+s}{iM^2-s}$,

$E_0(R\to 0) \sim \dfrac{\pi}{R}\,\dfrac{7}{60} \implies c_{UV} = \dfrac{7}{10}$

UV: tricritical Ising

$E_0(R\to \infty) \sim \dfrac{\pi}{R}\,\dfrac{1}{12} \implies c_{IR} = \dfrac{1}{2}$

IR: critical Ising

This is an example of asymptotic safety: the theory is nonrenormalizable but UV complete and flows to a UV fixed point. Asymptotic safety has been proposed as a solution to the nonrenormalizabity of Einstein gravity.

The NG string (E_{LC} theory ; $T\bar{T}$- deformed CFTs)

For this theory, we can use TBA to check that the S-matrix is compatible with the lightcone spectrum, which for the $\underbrace{\text{ground state}}_{N=\tilde{N}=0}$ reads

$$E_0 = \sqrt{\frac{R^2}{\ell_s^4} - \frac{4\pi^2}{\ell_s^2}\,\frac{(D-12)}{12}}$$

In this case we cannot take the UV limit $R \to 0$ since E_0 becomes complex.

From the string theory perspective (in $D=26$), this is the statement that there is a tachyon in the target space.

From the EFT perspective, we interpret this as a theory that does not flow to a UV fixed point despite having an S-matrix valid all the way to the UV. This behavior has been named "asymptotic fragility".

Hint of the absence of local observables: no local stress-energy tensor.

$E_0(R \to 0) \to$ complex $\Longrightarrow$ no $T_{\mu\nu}$ UV: theory of QG.

$E_0(R \to \infty) \sim \frac{\pi}{R} \frac{D-2}{6}$ $\Longrightarrow$ $c_{IR} = D-2$ IR: CFT of $D-2$ bosons