Aspects of holography and irrelevant deformations

Lecture 5:

consequences of the S-matrix

References:

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Last time we learned that unitarity, analiticity, and crossing symmetry imply that the S-matrix can be written as (here $\mathrm{Im}\, s > 0$)

$$S = \mathbb{1}\, e^{i\delta(s)} = \mathbb{1} \prod_j \frac{\mu_j + s}{\mu_j - s}\, e^{i P(s)}, \qquad P(-s) = -P(s), \qquad \mathrm{Im}\, \mu_j < 0$$

The simplest example of such an S-matrix compatible with locality ($P=0$) comes from the Goldstino $S_G = \mathbb{1}\, \frac{iM^2 - s}{iM^2 + s}$.

Today we will study some of the implications of the simplest S-matrix with $P \neq 0$, namely the S-matrix for the infinitely long bosonic string ($D=26$) or the theory with the lightcone spectrum (any D) where $S_{LC} = \mathbb{1}\, e^{i s \ell_s^2/4}$.

Consequences of the S-matrix

Let us consider two Gaussian wave-packets of different flavors.

$P_{L,R} = \dfrac{p^0 \mp p^1}{2}$

$x^j \,,\; p_j = (0, P_R)$ (right-mover) $\qquad x^i \,,\; p_i = (P_L, 0)$ (left-mover)

In the far past, the two-particle state (in the interaction picture) is

$$|in\rangle = \int_0^\infty dP_L \int_0^\infty dP_R \; f(P_L)\, f(P_R)\, \underbrace{\alpha^{i\dagger}(P_L)\, \tilde{\alpha}^{j\dagger}(P_R)}_{\text{creation operators of } x^i,\, x^j \ (i \neq j)} |0\rangle$$

Here $f(p_{L,R})$ is given by

$\bar{p}_L \equiv$ average momentum

$$f(p_L) = \frac{1}{\sqrt{2\pi}\,\Delta p_L}\, e^{-\frac{1}{2}\left(\frac{p_L - \bar{p}_L}{\Delta p_L}\right)^2}, \qquad \int_{-\infty}^{\infty} dp_L\, |f_L(p_L)|^2 = 1.$$

$\Delta p_L \equiv$ width of the distribution.

After the interaction, the two-particle state in the far future is

$$|out\rangle = \int_0^\infty dp_L \int_0^\infty dp_R\, f(p_L)\, f(p_R)\, \underbrace{e^{2i\delta(s)}}_{\equiv\, e^{i s l_s^2/4} \,=\, e^{i p_L p_R l_s^2}}\, \alpha^{i\dagger}(p_L)\, \tilde{\alpha}^{j\dagger}(p_R)\, |0\rangle$$

We see that the phase shift of the interaction entangles the left and right-moving modes. (Roughly $|in\rangle \sim \sum f_L\, |p_L\rangle \otimes \sum f_R\, |p_R\rangle \longrightarrow |out\rangle \sim \sum g_{LR}\, |p_L\rangle \otimes |p_R\rangle$)

(Also: this is a direct consequence of the energy dependence of the phase shift).

Let us consider the reduced density matrix of the left-movers

$$\rho_L(p_L, p_L') = \int_0^\infty dp_R\, f(p_L)\, f^*(p_L')\, |f(p_R)|^2\, e^{i p_R (p_L - p_L') l_s^2}$$

If we let $\bar{p}_R \gg \Delta p_R$ then $\int_0^\infty \to \int_{-\infty}^\infty$ and the reduced density matrix becomes

$$\rho_L(p_L, p_L') = f(p_L)\, f^*(p_L')\, e^{i \bar{p}_R (p_L - p_L') l_s^2 - \frac{1}{2} \Delta p_R^2 (p_L - p_L')^2 l_s^4}$$

We see that at high energies (i.e. $p_L, p_L', p_R \gg 1$) the reduced density matrix becomes diagonal and hence mixed. This means that the left-moving modes are highly entangled with the right-movers at high energies.

$\Rightarrow$ reminiscent of black hole creation / evaporation in a gravitational theory.

Note, however, that our theories do not feature particle production (i.e. the Hamiltonian is a function of the number operator), hence there can be no thermal radiation or black hole evaporation.

Nevertheless, from $\rho_L(p_L, p_L')$ we can compute the von Neumann entanglement entropy and obtain

$$S_{EE} = -\mathrm{Tr}(\rho \log \rho) = \log \underbrace{\Delta p_L \Delta p_R \ell_s^2}$$

the larger $\Delta p_L, \Delta p_R$ the larger the entropy

this makes sense: larger $\Delta p_L, \Delta p_R$ means that more left and right-movers become entangled

We can compute the von Neumann entropy via

$$S_{EE} = -\operatorname{Tr}(\rho \log \rho) = \lim_{n \to 1} -\frac{1}{n}\, \partial_n \operatorname{Tr} \rho^n$$

where n is an integer that we analitically continue to real values. Then

$$\operatorname{Tr} \rho^n = \int \prod_{i=1}^{n} d\rho_{Li}\, d\rho_{Ri}\ |\rho_{Li}\, \rho_{Ri}\rangle\langle \rho_{L(i+1)}\, \rho_{R(i+1)} | \rho_{L(i+1)}\, \rho_{R(i+1)}\rangle \ldots \langle \rho_{L(i+n-2)}\, \rho_{R(i+n-2)} | \rho_{L(i+n-1)}\, \rho_{R(i+n-1)}\rangle\langle \rho_{Li}\, \rho_{Ri}|$$

$$= \int \prod_{i=1}^{n} \frac{d\rho_{Li}\, d\rho_{Ri}}{2\pi\, \Delta\rho_L \Delta\rho_R}\, e^{i\rho_{Ri}(\rho_{Li} - \rho_{L(i+1)})\, l_s^2 - \frac{\rho_{Li}^2}{2\Delta\rho_L^2} - \frac{\rho_{Ri}^2}{2\Delta\rho_R^2}}$$

$$= \int \prod_{i=1}^{n} \frac{d\rho_{Li}}{(2\pi)^{1/2}\, \Delta\rho_L}\, e^{-i\frac{(\rho_{Li} - \rho_{L(i+1)})^2\, l_s^2\, \Delta\rho_R^2}{2} - \frac{\rho_{Li}^2}{2\Delta\rho_L^2}}$$

$$= \left\{ \cdot 2\alpha^n + \frac{1}{2^n}\left[\left(1 + 2\alpha - \sqrt{1+4\alpha}\right)^n + \left(1 + 2\alpha + \sqrt{1+4\alpha}\right)^n \right] \right\}^{-1/2}$$

where $\alpha \equiv \Delta\rho_L^2\, \Delta\rho_R^2\, l_s^4$. In the limit $\alpha \gg 1$ we then obtain

$$S_{EE} = \log \Delta\rho_L \Delta\rho_R\, l_s^2.$$

Spread of wave-packets

We now consider the density matrix in position space. For diagonal elements

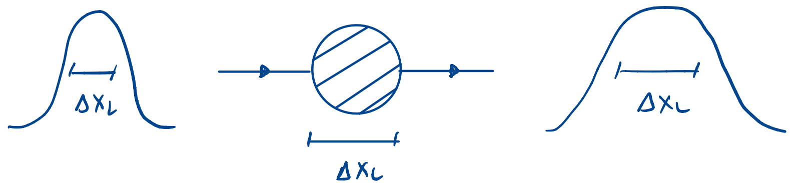
$$\rho(t+x_L, t+x_L) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dp_L \int_{-\infty}^{\infty} dp_L' \; \rho(p_L, p_L') \, e^{-i(p_L - p_L')(t+x_L)}$$

$$= \frac{1}{\sqrt{2\pi}\,\underbrace{\Delta x_L}} \exp\left[-\frac{(t + x_L - \bar{p}_R \ell_s^2)^2}{2\Delta x_L^2} \right]$$

$$\Delta x_L^2 = \frac{1}{4}\left(\frac{1}{\Delta p_L^2} + 4\ell_s^4 \Delta p_R^2 \right)$$

We see that after the interaction, the spread of the wave-packet has increased an amount proportional to the "coupling constant" ℓ_s^2 and the width (in momentum space) of the right-moving packet.

This means that any attempt to probe a scale of size ΔX_L will increase the size of the probe and ultimately fails!



Relatedly, we see from the definition of ΔX_L that

$$\Delta X_L \Delta X_R = \sqrt{(2\Delta P_L)^{-2} + l_s^4 \Delta P_R^2}\sqrt{(2\Delta P_R)^{-2} + l_s^4 \Delta P_L^2} \geq l_s^2 .$$

This means that we cannot probe scales smaller than l_s, which suggests the absence of local observables, as expected from a theory of QG!

Note that in generic nonrenormalizable EFTs there is a cutoff scale where the theory breaks down and new dofs are expected to make the theory local, e.g. the theory of the Goldstino.

What is surprising here is that the theory with spectrum $\mathcal{E}_{IC}$ has an S-matrix that is defined all the way to the UV, i.e. it is unitary, analytic, crossing symmetric.

Time delay

From the diagonal density matrix we also see that the outgoing wave-packet experiences a time delay $\Delta t = \bar{p}_R \ell_s^2$ that can be written in terms of the center-of-mass energy $E_{cm} = \sqrt{\bar{p}_L \bar{p}_R}$ as

$$\Delta t = \bar{p}_R \ell_s^2 \longrightarrow \frac{\ell_s^2}{2} E_{cm} .$$

<u>Comments</u>:

1. The time delay depends on the energy, which is characteristic of gravitational theories. In fact, in 2d models of black hole evaporation (e.g the CGHS model), the lifetime of a black hole of $\underbrace{\text{mass } M}_{T \sim M^0,\ S \sim M}$ is $\Delta t \propto M$

$\hookrightarrow$ 2d dilaton gravity + conformal matter

2. The scattering of multiple particles of different flavors (with momentum $\bar{p}^{(i)}_{L,R} = \bar{p}_{L,R}/N$) is also a phase shift:

$$2i\,\delta_{LC} = i\sum_i \frac{s_i\, l_s^2}{4} = \sum_i \bar{p}_L^{(i)}\, \bar{p}_R^{(i)}\, l_s^2 = \frac{\bar{p}_L \bar{p}_R}{N}\, l_s^2$$

Consequently, the time delay becomes

$$\Delta t = 2\,\delta_{LC}\,(\bar{p}_L/N)^{-1} = \bar{p}_R\, l_s^2 \longrightarrow \frac{l_s^2}{2} E_{cm} .$$

We interpret this as an equivalence principle: the time delay is universal (independent of flavor) and depends only on their energy.

3. If we change $\delta_{LC} \to -\delta_{LC}$, the S-matrix is no longer exponentially

bounded and the time delay becomes negative, i.e. the theory becomes acausal! Note that $\delta_{LC} \to -\delta_{LC}$ is equivalent to:

$$\underbrace{l_s^2 \to -l_s^2}$$

Recall that l_s^2 is our (dimensionful) "coupling constant"

↳ only l_s^2 appears in the action, spectrum, S-matrix

$$S = -\frac{1}{2}\int d^2\sigma\,(\partial_\alpha X^i)^2 - \frac{l_s^2}{8}\int d^2\sigma\,\left[(\partial_\alpha X^i\partial^\alpha X^i)^2 - 2(\partial_\alpha X^i\partial_\beta X^i\partial^\alpha X^j\partial^\beta X^j)\right] + \ldots$$

↓

we see that the sign of this coupling in the IR is fixed by constraints from the UV → (recall David Simmons-Duffin's talk).

Summary:

We have described a class of (integrable) theories characterized by the spectrum and $(2 \to 2)$ S-matrix:

$$E_{cc} = \sqrt{\frac{4\pi^2 (N - \tilde{N})^2}{R^2} + \frac{R^2}{\ell_s^4} + \frac{4\pi}{\ell_s^2}\left(N + \tilde{N} - \frac{D-2}{12}\right)} \qquad \Longleftrightarrow \qquad S = \mathbb{1}\, e^{is\ell_s^2/4}$$

In $D=26$ this is the Nambu-Goto string, for which we can derive E_{cc} by lightcone quantization (with some assumptions).

For any D we can interpret these theories as simple toy models of quantum gravity where $e^{is\ell_s^2/4}$ is the gravitational dressing of the S-matrix of

D-2 free scalars for which $S = \mathbb{1}$.

More generally (for nonintegrable QFTs) we can dress any S-matrix S_{QFT} by an analog of the Castillejo-Dalitz-Dyson factor:

$$S_{\text{dressed}} = e^{i \ell_s^2/4 \sum_{i<j} p_i * p_j} \; S_{QFT}$$

where $p_i * p_j \equiv \epsilon_{\alpha\beta} \, p_i^\alpha \, p_j^\beta$ and the particles are ordered according to their rapidities $\theta_1 < \theta_2 < \dots < \theta_n$ where

$$\left.\begin{aligned} p_i^0 &= m \cosh \theta_i \\ p_i^1 &= m \sinh \theta_i \end{aligned}\right\}$$

s.t. amplitudes depend only on the difference of rapidities

We now have a more precise and universal definition of these theories:

They're CFTs (or QFTs) deformed instantaneously by the irrelevant

$T\bar{T}$ operator:

$$\frac{\partial S}{\partial \mu} = -4 \int d^2x \; T\bar{T}$$

$$T\bar{T} \equiv T_{++} T_{--} - T_{+-} T_{-+}$$

$$\mu \propto l_s^2$$

UV: theory of QG.

IR: CFT of D-2 bosons