Aspects of holography and irrelevant deformations

Lecture 6:

the $T\bar{T}$ operator

References:

0401196, 1608.05499

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We are interested in computing the expectation value or one-point function of an operator built from the stress-energy tensor $T_{\mu\nu}$ of a QFT.

One-point functions incorporate information about the vacuum state of the theory, e.g. $\langle T_{zz} \rangle = \langle T_{\bar{z}\bar{z}} \rangle = -c/24$ for a CFT on the cylinder. They are usually nonperturbative quantities and no general approach to their evaluation is known.

In this lecture we will show that (given some assumptions):

1. The operator $T\bar{T} \equiv 4\pi^2 \lim_{z \to z'} \left[T_{zz}(z) T_{\bar{z}\bar{z}}(z') - T_{z\bar{z}}(z) T_{z\bar{z}}(z') \right]$ is well-defined (i.e. finite) in any QFT.
2. The expectation value of this operator is constant and factorizes, meaning that $\langle T\bar{T} \rangle = 4\pi^2 \left(\langle T_{zz} \rangle \langle T_{\bar{z}\bar{z}} \rangle - \langle T_{z\bar{z}} \rangle^2 \right)$.
3. Perturbations by $T\bar{T}$ (and its generalizations) preserve integrability.

Assumptions : we must make several assumptions about the local and global properties of the QFTs we are interested in. For convenience we define :

$$T \equiv -2\pi T_{zz} \qquad \bar{T} \equiv -2\pi T_{\bar{z}\bar{z}} \qquad \Theta \equiv 2\pi T_{z\bar{z}}\,, \qquad (z,\bar{z}) = (x+iy,\, x-iy)$$

I. Local translational and rotational symmetry

$$\partial_\mu T^\mu{}_\nu = 0 \qquad\qquad T_{\mu\nu} = T_{\nu\mu}$$

⇝ we can (and will) relax this later

We can write the conservation equation in a more convenient way:

$$\partial_\mu T^\mu{}_\nu = 0 \quad \Rightarrow \quad \underbrace{\begin{aligned} &\partial_z T_{\bar{z}z} + \partial_{\bar{z}} T_{zz} = 0 \\ &\partial_{\bar{z}} T_{z\bar{z}} + \partial_z T_{\bar{z}\bar{z}} = 0 \end{aligned}} \quad \begin{aligned} &\Rightarrow \quad \bar{\partial} T = \partial \Theta \\ &\Rightarrow \quad \partial \bar{T} = \bar{\partial} \Theta \end{aligned}$$

here we used $ds^2 = dz\, d\bar{z} \Rightarrow g_{zz} = g_{\bar{z}\bar{z}} = g^{zz} = g^{\bar{z}\bar{z}} = 0$

$$g_{z\bar{z}} = \frac{1}{2}\,, \qquad g^{z\bar{z}} = 2.$$

II. Local interactions

We assume all the interactions are local such that, at long enough distances, the connected part of any two-point function vanishes:

$$\lim_{\alpha \to \infty} \langle O(z + \alpha \cdot \vec{x}) O'(z) \rangle = \langle O(z) \rangle \langle O'(z) \rangle \,, \qquad O, O' = \text{arbitrary operators}$$

$\vec{x}$ is some direction on the (Euclidean) plane

III. Global translational symmetry

We assume that the one and two-point functions of any field $O(z)$ satisfy:

$$\langle O(z) \rangle = \text{constant}, \qquad \langle O(z) O(z') \rangle = f(z - z')$$

This assumption is related but different from local translational symmetry. Here we are assuming the translational symmetry is not spontaneously broken* – we assume it's a global symmetry of the theory - in the sense that (here for convenience we assume $\mathcal{O}(z)$ is chiral).

$$\langle \mathcal{O}(z) \rangle = \langle U_L^\dagger \mathcal{O}(z_0) U_L \rangle = \langle \mathcal{O}(z_0) \rangle \quad \Rightarrow \quad \langle \mathcal{O}(z) \rangle \text{ is constant}$$

$U_L \equiv e^{iH_L(z-z_0)}$, $U_L |0\rangle \to$ not necessarily 0 (e.g. for a QFT on the cylinder) but its effect cancels that of $\langle 0| U^\dagger$

(here $H_L = H+P$ and $H_R = H-P$ generate translations along $z, \bar{z}$)

$$\langle \mathcal{O}(z)\, \mathcal{O}'(z') \rangle = \langle U^\dagger \mathcal{O}(z_0)\, \mathcal{O}(z'-z-z_0)\, U \rangle = \langle \mathcal{O}(z_0)\, \mathcal{O}(z'-z-z_0) \rangle \quad \Rightarrow \quad \langle \mathcal{O}(z)\, \mathcal{O}'(z') \rangle = f(z-z')$$

Assumptions II and III above imply that the underlying geometry is either a plane or an infinitely long cylinder. We don't have a general (unambiguous) definition of the $T\bar{T}$ operator in other spacetimes.

We will now show that $C \equiv \langle T(z)T(z') \rangle - \langle \Theta(z)\, \Theta(z') \rangle$ factorizes and is a constant in any QFT satisfying the assumptions above.

Factorizability of $T\bar{T}$

First, let us show that C is constant (for convenience we drop the $\bar{z}$ dependence of T, $\bar{T}$, and Θ)

$$
\begin{aligned}
\partial_{\bar z} C &= \langle \partial_{\bar z} T(z)\, \bar{T}(z') \rangle - \partial_{\bar z} \langle \Theta(z)\Theta(z') \rangle && \text{use I on the first term} \\
&= \langle \partial_{z} \Theta(z)\, \bar{T}(z') \rangle - \partial_{\bar z} \langle \Theta(z)\Theta(z') \rangle \\
&= \partial_{z} \langle \Theta(z)\, \bar{T}(z') \rangle - \partial_{\bar z} \langle \Theta(z)\Theta(z') \rangle && \text{use III on both terms} \\
&= -\partial_{z'} \langle \Theta(z)\, \bar{T}(z') \rangle + \partial_{\bar z'} \langle \Theta(z)\Theta(z') \rangle \\
&= -\langle \Theta(z) \underbrace{\left[\partial_{z'} \bar{T}(z') - \partial_{\bar z'} \Theta(z') \right]}_{=0} \rangle \\
&= 0
\end{aligned}
$$

Similarly $\partial_z C = 0$. Then using III once again we find that local and global translational invariance implies that

$$\partial_z C = \partial_{z'} C = \partial_{\bar z} C = \partial_{\bar z'} C = 0 \quad \Rightarrow \quad C \text{ is a constant.}$$

Since C is a constant we can put z and z' at any location. In particular, we can choose z and z' to be infinitely separated such that (by assumption 3)

$$C = \langle T(z)\bar{T}(z')\rangle - \langle\Theta(z)\Theta(z')\rangle = \langle T(z)\rangle\langle\bar{T}(z')\rangle - \langle\Theta(z)\rangle\langle\Theta(z')\rangle$$

$$= \underbrace{\langle T\rangle\langle\bar{T}\rangle - \langle\Theta\rangle^2}_{\text{by global translational invariance these one-point functions are constant}}$$

Note that $T(z)\bar{T}(z') - \Theta(z)\Theta(z')$ is not yet the $T\bar{T}$ operator, which is defined at coincident points $z \to z'$.

Also note that the combination of operators in the definition of C is necessary to make C constant. This combination also removes the divergences in the two-point functions as we take $z \to z'$. (A side comment: in a CFT we don't need to worry about possible short distance divergences in the definition of $T\bar{T}(z)$. The reason is that in a CFT, this quantity is finite and a descendant of the vacuum, namely $T\bar{T}(z) = \lim_{z' \to z} \overline{T(z)\bar{T}(z')} = L_{-2}\bar{L}_{-2}\mathbb{1}$).

The $T\bar{T}$ operator

We now show that the $T\bar{T}$ operator

$$T\bar{T}(z') = \lim_{z \to z'} \left[T(z)\bar{T}(z') - \Theta(z)\Theta(z') \right]$$

is well-defined (free of divergences) in any QFT satisfying assumptions I - III up to total derivative terms that don't affect its expectation value.

In order to see this let us introduce the following operator product expansions (OPEs)

$$\begin{pmatrix} T(z)\,\bar{T}(z') & T(z)\,\Theta(z') \\ \Theta(z)\,\bar{T}(z') & \Theta(z)\,\Theta(z') \end{pmatrix} = \sum_i \begin{pmatrix} D_i(z-z') & A_i(z-z') \\ B_i(z-z') & C_i(z-z') \end{pmatrix} \mathcal{O}_i(z')$$

where we sum over all the local operators $\mathcal{O}_i$ of the QFT.

Using the conservation of the stress tensor we can show that

$$\partial_{\bar z}\left[\,T(z)\,\bar T(z') - \Theta(z)\,\Theta(z')\,\right] = \underbrace{\partial_{\bar z}\,T(z)}_{\partial_z \Theta(z)}\,\bar T(z') - \partial_{\bar z}\,\Theta(z)\,\Theta(z') + \Theta(z)\underbrace{\left[\,\partial_{z'}\bar T(z') - \partial_{\bar z'}\,\Theta(z')\,\right]}_{0}$$

$$= (\partial_z + \partial_{z'})\,\Theta(z)\,\bar T(z') - (\partial_{\bar z} + \partial_{\bar z'})\,\Theta(z)\,\Theta(z')$$

Similarly, it's not difficult to show that

$$\partial_z\left[\,T(z)\,\bar T(z') - \Theta(z)\,\Theta(z')\,\right] = (\partial_z + \partial_{z'})\,T(z)\,\bar T(z') - (\partial_{\bar z} + \partial_{\bar z'})\,T(z)\,\Theta(z')$$

Note that both of these expressions feature $\partial_z + \partial_{z'}$ and $\partial_{\bar z} + \partial_{\bar z'}$ derivatives which satisfy $(\partial_z + \partial_{z'})(z - z') = 0$ and $(\partial_{\bar z} + \partial_{\bar z'})(\bar z - \bar z') = 0$. Consequently,

$$\partial_{\bar z}\left[\,T(z)\,\bar T(z') - \Theta(z)\,\Theta(z')\,\right] = \sum_i \left[\,D_i(z-z')\,\partial_{z'}\,\mathcal{O}_i(z') - A_i(z-z')\,\partial_{\bar z'}\,\mathcal{O}_i(z')\,\right]$$

$$\partial_z\left[\,T(z)\,\bar T(z') - \Theta(z)\,\Theta(z')\,\right] = \sum_i \left[\,B_i(z-z')\,\partial_{z'}\,\mathcal{O}_i(z') - C_i(z-z')\,\partial_{\bar z'}\,\mathcal{O}_i(z')\,\right]$$

From these equations we learn that

$$T(z)\bar{T}(z') - \Theta(z)\Theta(z') = \Sigma_i \underbrace{\left[D_i(z-z') - C_i(z-z') \right]} O_i(z')$$

$$= \Sigma_i \, F_i(z-z') \, O_i(z')$$

$$= \text{constant} \; \underbrace{O_{T\bar{T}}(z')}_{\text{the } T\bar{T} \text{ operator}} + \Sigma_i \left[\tilde{F}_i(z-z') \, \partial_{z'} O_i(z') + \bar{\tilde{F}}_i(z-z') \, \partial_{\bar{z}'} O_i(z') \right]$$

may diverge as $z \to z'$

Note that the structure of the OPE holds for any $X(z)Y(z')$ where $\partial_z X(z)Y(z')$ and $\partial_{\bar{z}} X(z) Y(z')$ feature only descendant $\partial_z O_i(z)$ and $\partial_{\bar{z}} O_i(\bar{z})$ terms in the OPE.

The $T\bar{T}$ operator is then defined as

$$T(z)\bar{T}(z') - \Theta(z)\Theta(z') = T\bar{T}(z') + \text{derivative terms}$$

By assumption III $\langle O_i(z) \rangle$ is constant and hence $\langle \partial_z O_i(z) \rangle = \langle \partial_{\bar{z}} O_i(z) \rangle = 0$ for any field O_i. Thus, taking the expectation value we find

Thus we find that the expectation value of the $T\bar{T}$ operator factorizes in any QFT satisfying assumptions I - III, namely

$$\langle T\bar{T} \rangle = \lim_{z \to z'} \left[\langle T(z)\bar{T}(z') \rangle - \langle \Theta(z)\, \Theta(z') \rangle \right] = \langle T \rangle \langle \bar{T} \rangle - \langle \Theta \rangle^2$$

This feature of the $T\bar{T}$ operator holds also for more general states. Indeed for nondegenerate eigenstates $|n\rangle$ of the energy and momentum we have

$$\boxed{\langle n | T\bar{T} | n \rangle = \lim_{z \to z'} \left[\langle n | T(z)\, \bar{T}(z') | n \rangle - \langle n | \Theta(z)\, \Theta(z') | n \rangle \right] = \langle n | T | n \rangle \langle n | \bar{T} | n \rangle - \langle n | \Theta | n \rangle^2}$$

Let us quickly show how to obtain this result. We first note that assumptions I and III (local and global translation invariance) still hold. Assumption III follows from (we assume $\Theta(z)$ is chiral for convenience)

$$\langle n | \Theta(z) | n \rangle = \langle n | U_L^{\dagger}\, \Theta(z_0)\, U_L | n \rangle = \langle n | \Theta(z_0) | n \rangle \quad \Rightarrow \quad \langle n | \Theta(z) | n \rangle \text{ is constant}$$

since $|n\rangle$ is an eigenstate of energy and momentum

Similarly $\langle n| \mathcal{O}(z)\, \mathcal{O}(z')|n\rangle = \langle n| U_L^\dagger \mathcal{O}(z_0)\, \mathcal{O}(z'-z-z_0)\, U_L |n\rangle = \langle n| \mathcal{O}(z_0)\, \mathcal{O}(z'-z-z_0)|n\rangle$ so that $\langle n|\mathcal{O}(z)\,\mathcal{O}(z')|n\rangle = f(z-z')$ for some function f.

Let $C_n \equiv \langle n|T(z)\,\bar{T}(z')|n\rangle - \langle n|\Theta(z)\,\bar{\Theta}(z')|n\rangle$. Then, assumptions I and III imply that C_n is a constant.

Now we cannot use assumption II (local interactions) to argue that C_n factorizes. This is because $\langle n|\mathcal{O}(z)\,\mathcal{O}(z')|n\rangle$ can be understood as a four-point function with two $\mathcal{O}_n$ operators inserted at $\rho=0$ and $\rho\to\infty$ (here $z=\rho e^{i\phi}$).

Nevertheless, since $|n\rangle$ is an energy/momentum eigenstate we can write:

$$C_n = \sum_{n'} \langle n|T(z)|n'\rangle \underbrace{\langle n'|\bar{T}(z')|n\rangle}_{} + \langle n|\Theta(z)|n'\rangle\langle n'|\bar{\Theta}(z')|n\rangle$$

$$= \langle n'|U_L^\dagger U_R^\dagger\, \bar{T}(z)\, U_L U_R|n\rangle$$

$$= \langle n'|\bar{T}(z)|n\rangle\, e^{i[\epsilon_L(n)-\epsilon_L(n')](z'-z) + i[\epsilon_R(n)-\epsilon_R(n')](\bar{z}'-\bar{z})}$$

also use $\bar{\Theta}(z') = U_L^\dagger U_R^\dagger \bar{\Theta}(z)\, U_R U_L$

$$C_n = \sum_{n'} \left[\langle n | T(z) | n' \rangle \langle n' | \bar{T}(\bar{z}) | n \rangle + \langle n | \Theta(z) | n' \rangle \langle n' | \Theta(z) | n \rangle \right] e^{i[E_L(n) - E_L(n')](z'-z) + i(E_R(n) - E_R(n')](\bar{z}'-\bar{z})}$$

Since C_n is constant, we see that all the terms with $n \neq n$ must cancel. Thus, assuming that $|n\rangle$ is nondegenerate, we obtain

$$C_n = \langle n | T(z) | n \rangle \langle n | \bar{T}(\bar{z}) | n \rangle - \langle n | \Theta(z) | n \rangle \langle n | \Theta(z) | n \rangle$$

Finally, since the OPE argument applies to any state, the $T\bar{T}$ operator satisfies

$$\langle n | T\bar{T} | n \rangle = \langle n | T | n \rangle \langle n | \bar{T} | n \rangle - \langle n | \Theta | n \rangle^2$$