Aspects of holography and irrelevant deformations

Lecture 7:

current deformations and integrability

References:

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The $T\bar{T}$ operator

In the previous lecture we showed that any $d=2$ QFT with $\underbrace{\text{local + global}}$ translational invariance has a finite operator satisfying (QFT is defined on the plane/cylinder)

$$\langle n | T\bar{T} | n \rangle = \lim_{z \to z'} \langle n | T(z) \bar{T}(z') - \Theta(z)\Theta(z') | n \rangle = \langle n | T | n \rangle \langle n | \bar{T} | n \rangle - \langle n | \Theta | n \rangle^2$$

where $|n\rangle$ is a nondegenerate eigenstate of the energy and the momentum. In particular this expression holds for the vacuum when $|n\rangle \to |0\rangle$.

Generalization to other currents

The result above holds universally for any local QFT invariant under translations.

The main feature of the $T\bar{T}$ operator is that it's built from a particular combination of two conserved currents which guarantees $\langle T\bar{T} \rangle$ is free of divergences and that it factorizes into products of one-point functions.

Let us consider a theory with additional conserved currents. We introduce the following notation:

$$\underbrace{T_{s+1} \equiv \overset{\text{spin } s+1}{T_{s+1,0}}, \qquad \Theta_{s-1} \equiv \overset{\text{spin } s-1}{\Theta_{s,1}}}_{\text{weight } s+1}, \qquad \underbrace{\bar{T}_{s+1} \equiv T_{0,s+1}, \qquad \bar{\Theta}_{s-1} \equiv \bar{\Theta}_{1,s}}_{\text{weight } s+1}$$

where $A_{\alpha,\beta}$ has left and right-moving energies $\epsilon_L = \alpha$, $\epsilon_R = \beta$, such that

$$\text{weight } \Delta = \epsilon_L + \epsilon_R = \alpha + \beta, \qquad \text{spin } s = \epsilon_L - \epsilon_R = \alpha - \beta$$

These operators satisfy

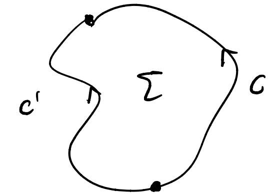
$$\partial_{\bar z} T_{s+1}(z) = \partial_z \Theta_{s-1}(z), \qquad \partial_z \bar{T}_{s+1}(z) = \partial_{\bar z} \bar{\Theta}_{s-1}(z).$$

In differential notation we have the following conserved currents

$$\begin{aligned} j_{s+1} &= T_{s+1}\, dz - \Theta_{s-1}\, d\bar z \\ \bar{J}_{s+1} &= \bar{\Theta}_{s-1}\, dz - \bar{T}_{s+1}\, d\bar z \end{aligned} \quad \Rightarrow \quad \begin{aligned} * j_{s+1} &= T_{s+1}\, dz + \Theta_{s-1}\, d\bar z, \qquad & d * j_{s+1} = 0 \\ * \bar{J}_{s+1} &= \bar{\Theta}_{s-1}\, dz + \bar{T}_{s-1}\, d\bar z, \qquad & d * \bar{J}_{s+1} = 0 \end{aligned}$$

These currents lead to conserved charges

$$0 = \int_\Sigma d * j \quad \Rightarrow \quad \begin{aligned} P_s &= \int_c * j_{s+1} \\ \bar{P}_s &= \int_c * \bar{j}_{s+1} \end{aligned}$$



$c \cup (-c') = \partial\Sigma$

For each pair of currents of spin $\pm(s+1)$ we can define the following Lorentz scalar

$$X_s \equiv \lim_{z \to z'} \left[T_{s+1}(z) \bar{T}_{s+1}(z') - \Theta_{s-1}(z) \bar{\Theta}_{s-1}(z') \right].$$

Alternatively we can write X_s in terms of $j, \bar{j}$ as follows

$$X_s \, dz \wedge d\bar{z} = \lim_{z \to z'} j_s(z) \wedge \bar{j}_s(\bar{z}).$$

In this notation $X_1 = T\bar{T}$ is one in an infinite family of operators with weight $\Delta = 2(s+1)$. Note that all of these operators are irrelevant, i.e. $\Delta > 2$. Hence, perturbing a QFT by any of the X_s changes its UV behavior.

Following the same steps as before we can show that translation invariance implies

$$\langle n| T_{s+1}(z)\, \bar{T}_{s+1}(z')|n\rangle - \langle n|\Theta_{s-1}(z)\, \bar{\Theta}_{s-1}(z')|n\rangle = \langle n|T_{s+1}(z)|n\rangle\langle n|\bar{T}_{s+1}(z)|n\rangle - \langle n|\Theta_{s-1}(z)|n\rangle\langle n|\bar{\Theta}_{s-1}(z)|n\rangle$$

$$T_{s+1}(z)\, \bar{T}_{s+1}(z') - \Theta_{s-1}(z)\, \bar{\Theta}_{s-1}(z') = X_s + \text{derivatives}$$

Consequently $\langle n| X_s |n\rangle$ factorizes and is given by

$$\langle n| X_s |n\rangle = \langle n|T_{s+1}(z)|n\rangle\langle n|\bar{T}_{s+1}(z)|n\rangle - \langle n|\Theta_{s-1}(z)|n\rangle\langle n|\bar{\Theta}_{s-1}(z)|n\rangle$$

<u>Integrability</u>

An integrable QFT is characterized by an infinite number of commuting conserved charges P_s, $\bar{P}_s$ such that

$$[P_s, P_{s'}] = [P_s, \bar{P}_{s'}] = [\bar{P}_s, \bar{P}_{s'}] = 0$$

A generic deformation of an integrable QFT (IQFT) breaks integrability.

Interestingly, deforming an IQFT by any of the X_s operators preserves its integrability (at least infinitesimally).

$$S_{IQF} \to S_{IQFT} + \delta S \,, \qquad \delta S = \mu \int d^2w \, X_s \qquad [\mu] = (\text{length})^{2s+2}$$

In order to see this we first recall the definition of P_s and $\bar{P}_s$:

$$P_s = \int_c * j = \int_c (T_{s+1} dz + \Theta_{s-1} d\bar{z}) \,, \qquad \bar{P}_s = \int_c * \bar{j} = \int_c (\bar{\Theta}_{s-1} dz + \bar{T}_{s+1} d\bar{z})$$

The fact that all of the P_s currents commute implies that (similar expressions exist for $\bar{P}_s$)

$$[P_\sigma, T_{s+1}(z)] = \partial_z A_{\sigma,s}(z) \,, \qquad [P_\sigma, \Theta_{s-1}(z)] = \partial_{\bar{z}} A_{\sigma,s}(z)$$

$$[P_\sigma, \bar{T}_{s+1}(z)] = \partial_{\bar{z}} B_{\sigma,s}(z) \,, \qquad [P_\sigma, \bar{\Theta}_{s-1}(z)] = \partial_z B_{\sigma,s}(z)$$

Using these eqs. we can show that (here we drop the coordinate dependence for convenience)

$$[P_\sigma, X_s] = \lim_{z \to z'} \underbrace{[P_\sigma, T_{s+1}]\bar{T}_{s+1} - [P_\sigma, \Theta_{s-1}]\bar{\Theta}_{s-1}}_{I_1} + \underbrace{T_{s+1}[P_\sigma, \bar{T}_{s+1}] - \Theta_{s-1}[P_\sigma, \bar{\Theta}_{s-1}]}_{I_2}$$

$$I_1 = \partial_z A_{\sigma,s} \bar{T}_{s+1} - \partial_{\bar{z}} A_{\sigma,s} \bar{\theta}_{s-1} + A_{\sigma,s} \underbrace{\left(\partial_{z'} \bar{T}_{s+1} - \partial_{\bar{z}'} \bar{\theta}_{s-1} \right)}_{0}$$

$$= (\partial_z + \partial_{z'}) (A_{\sigma,s} \bar{T}_{s+1}) - (\partial_{\bar{z}} + \partial_{\bar{z}'}) (A_{\sigma,s} \bar{\theta}_{s-1})$$

Similarly, we can show that

$$I_2 = (\partial_{\bar{z}} + \partial_{\bar{z}'}) (B_{\sigma,s} T_{s+1}) - (\partial_z + \partial_{z'}) (B_{\sigma,s} \theta_{s-1})$$

As a result we can write the $[\rho_\sigma, X_s]$ commutator as follows:

$$[\rho_\sigma, X_s] = \partial_{\bar{z}} \underbrace{\hat{T}_{\sigma+1,s} - \partial_z \hat{\theta}_{\sigma-1,s}}$$

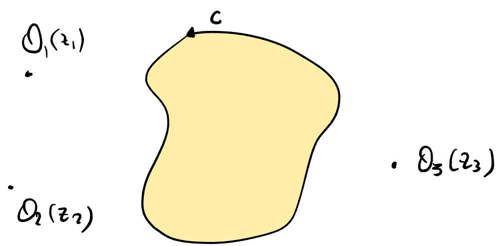
these operators can be determined

explicitly from I_1 and I_2 above

Our strategy will be to show that $d * j = 0$ within correlation functions, namely that the following quantity vanishes

$$\langle \Pi_i \, \mathcal{O}_i(z_i) \, d * j \rangle = 0 \quad \Rightarrow \quad J \equiv \langle \Pi_i \, \mathcal{O}_i(z_i) \oint_C * j \rangle = 0$$

Here $\oint_C$ denotes a closed contour and we assume operators $O_i(z_i)$ are located outside the region enclosed by the contour,



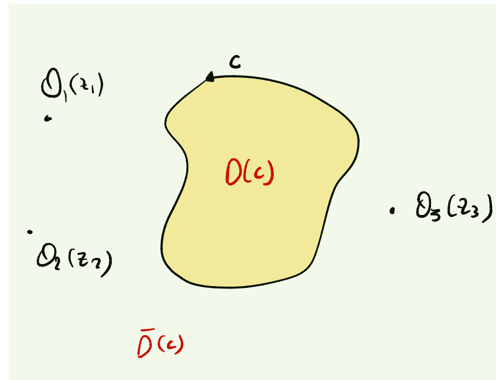
Before the deformation $J = \langle \prod_i O_i(z_i) \oint_C *j \rangle = 0$. After the deformation $\delta S = \mu \int d^2w \, X_S$, the correlator becomes

$$\delta J = \underbrace{\langle \prod_i \delta O_i(z_i) \oint_C *j \rangle}_{\text{vanishes at zeroth order}} + \langle \prod_i O_i(z_i) \oint_C * \delta j \rangle - \mu \int d^2w \, \langle X_S(w) \prod_i O_i(z_i) \oint_C *j \rangle + O(\mu^2)$$

we have the freedom to choose δj to make $\delta J = 0$.

The integral in the last term can be written as

$$\int d^2w \, \langle X_S(w) \, \Pi_i \, O_i(z_i) \oint_C * j \rangle = \int_{\bar{D}(c)} d^2w \, \langle \ldots \rangle + \int_{D(c)} d^2w \, \langle \ldots \rangle$$



In the first term the $X_S(w)$ operator is outside the region bounded by the contour c, hence $\oint_c * j = 0$ implies

$$\int_{\bar{D}(c)} d^2w \, \langle X_S(w) \, \Pi_i \, O_i(z_i) \oint_C * j \rangle = 0.$$

In the second term the $X_S(w)$ operator is inside the contour c, so we can write

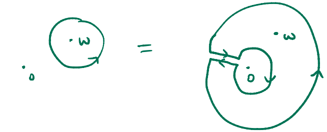
$$\int_{D(c)} d^2w \, \langle X_s(w) \prod_i \mathcal{O}_i(z_i) \oint_c * j \rangle = \int_{D(c)} d^2w \, \langle \prod_i \mathcal{O}_i(z_i) \underbrace{\oint_c X_s(w) * j}\, \rangle$$

$$= \oint_c X_s(w)\, T_{\sigma+1}\, dz + X_s(w)\, \Theta_{\sigma-1}\, d\bar{z}$$

$$= [\, P_\sigma , X_s(w) \,] \longrightarrow$$

recall that if $A = \oint a\, dz$ then

$$\oint_w dz\, a(z)\, b(w) = \oint_{C_1} dz\, a(z)\, b(w) - \oint_{C_2} dz\, b(w)\, a(z)$$



from $[P_\sigma, P_s] = 0$

$$= \partial_{\bar{z}} \hat{T}_{\sigma+1,s} - \partial_z \hat{\Theta}_{\sigma-1,s}$$

We can now write $\partial_{\bar{z}} \hat{T}_{\sigma+1,s} - \partial_z \hat{\Theta}_{\sigma-1,s} \equiv d * \hat{j}$ where $\hat{j} = \hat{T}_{\sigma+1,s}\, dz - \hat{\Theta}_{\sigma-1,s}\, d\bar{z}$. We thus have

$$\int_{D(c)} d^2w \, \langle X_s(w) \prod_i \mathcal{O}_i(z_i) \oint_c * j \rangle = \langle \prod_i \mathcal{O}_i(z_i) \int_{D(c)} d^2w \; d * \hat{j} \rangle$$

$$= \langle \prod_i \mathcal{O}_i(z_i) \oint_c * \hat{j} \rangle$$

Altogether, to linear order in the deformation we find

$$\delta J = \langle \prod_i \mathcal{O}_i(z_i) \oint_c * (\delta j - \mu \hat{j}) \rangle$$

Thus we can guarantee that each of the conserved currents $j = T_{\sigma+1}\, dz - \Theta_{s-1}\, d\bar{z}$ of the original IQFT is conserved after the deformation by any of the irrelevant X_s operators provided that

$$\delta j = \delta T_{s+1}\, dz - \delta \Theta_{s-1}\, d\bar{z} = \mu\, \hat{T}_{\sigma+1,s}\, dz - \mu\, \Theta_{\sigma-1,s}\, d\bar{z} = \mu\, \hat{j}.$$

<u>Comments</u>:

- The expression above for δj determines how the currents flow after the deformation.
- Relatedly, we need to specify how the theory is deformed beyond linear order in μ
- We have not shown that $[P_\sigma, P_s] = 0$ after the deformation. This is still expected to be the case since $Q_{\sigma+s} \propto [P_\sigma, P_s]$ must be either a new conserved charge of spin $\sigma+s$ or an old charge $Q_{\sigma+s} \to P_{\sigma+\alpha}$. New conserved currents or a new algebraic structure are not expected to appear for infinitesimal μ such that $[P_\sigma, P_s] = 0$.

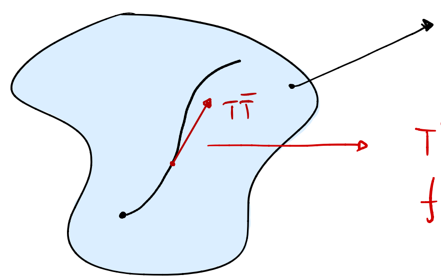
The $T\bar{T}$ flow

Let us now return to the simplest (Lorentz invariant) deformation: $X_1 = T\bar{T}$.

The $T\bar{T}$ deformation is an irrelevant deformation defined by

$$\delta S = -\frac{\mu}{\pi^2}\int d^2z \; T\bar{T}$$

at every point along the deformation



space of 2d QFTs

$T\bar{T}$ is the "vector" tangent to the flow

In other words, the $T\bar{T}$ (and related) deformations are defined by

$$\frac{\partial S}{\partial \mu} = -\frac{1}{\pi^2}\int d^2z \,(T\bar{T})_\mu$$

where $(T\bar{T})_\mu$ is the instantaneous $T\bar{T}$ operator of the deformed theory.

This differential equation implies that the classical action is deformed (in principle) by an infinite number of irrelevant operators. However we can still, and generically do have, additional terms not determined by this equation (more on this later).