Aspects of holography and irrelevant deformations

Lecture 8:

the $T\bar{T}$ spectrum

References:

1608.05499, 1608.05534

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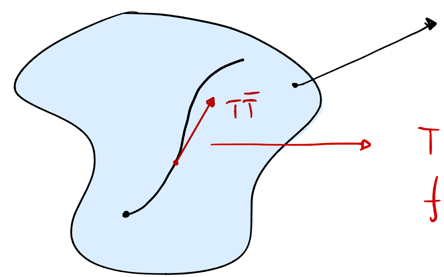
The $T\bar{T}$ flow

The $T\bar{T}$ deformation is an irrelevant deformation defined by

$$\frac{\partial S}{\partial \mu} = -\frac{1}{\pi^2}\int d^2z\, (T\bar{T})_\mu$$

at every point along the deformation



$T\bar{T}$ is the "vector" tangent to the flow

where $(T\bar{T})_\mu$ is the instantaneous $T\bar{T}$ operator of the deformed theory.

This differential equation implies that the classical action is deformed (in principle) by an infinite number of irrelevant operators. However we generically have additional terms not determined by this equation.

The $T\bar{T}$ deformation of the free boson

Let us illustrate the meaning of the instantaneous $T\bar{T}$ deformation with an explicit example. We take the undeformed action to be

$$S(0) = \int dz\, d\bar{z}\, \mathcal{L}(0), \qquad \mathcal{L}(0) = \partial\phi\, \bar{\partial}\phi \qquad (\partial \equiv \partial_z, \quad \bar{\partial} \equiv \partial_{\bar{z}})$$

For convenience we define $\tau = T/\pi$, $\bar{\tau} = \bar{T}/\pi$, $\omega = \Theta/\pi$, such that

$$\partial_\mu \mathcal{L}(\mu) = -(\tau\bar{\tau})_\mu = -\left[\tau(\mu)\bar{\tau}(\mu) - \omega(\mu)^2\right]$$

The components of the stress tensor in the deformed theory are defined by

$$\tau = -\frac{\partial\mathcal{L}(\mu)}{\partial(\bar{\partial}\phi)}\partial\phi, \qquad \bar{\tau} = -\frac{\partial\mathcal{L}(\mu)}{\partial(\partial\phi)}\bar{\partial}\phi, \qquad \omega = \frac{1}{2}\left[\frac{\partial\mathcal{L}(\mu)}{\partial(\partial\phi)}\partial\phi + \frac{\partial\mathcal{L}(\mu)}{\partial(\bar{\partial}\phi)}\bar{\partial}\phi - 2\mathcal{L}(\mu)\right]$$

Plugging these into the differential equation above then yields

$$\partial_\mu \mathcal{L}(\mu) = -\frac{\partial\mathcal{L}(\mu)}{\partial(\bar{\partial}\phi)}\frac{\partial\mathcal{L}(\mu)}{\partial(\partial\phi)}\partial\phi\,\bar{\partial}\phi + \frac{1}{4}\left[\frac{\partial\mathcal{L}(\mu)}{\partial(\partial\phi)}\partial\phi + \frac{\partial\mathcal{L}(\mu)}{\partial(\bar{\partial}\phi)}\bar{\partial}\phi - 2\mathcal{L}(\mu)\right]^2$$

We can solve this equation perturbatively by letting

$$\mathcal{L}(\mu) = L^{(0)} + \mu L^{(1)} + \dots = \sum_{i=0}^{\infty} \mu^{i} L^{(i)}$$

independent of μ

In this way we can write the ingredients in $\partial_\mu \mathcal{L}(\mu) = -(\tau\bar{\tau})_\mu$ as follows

$$\partial_\mu \mathcal{L}(\mu) = \sum_{i=0}^{\infty} i\,\mu^{i-1} L^{(i)} = \sum_{i=0}^{\infty} (i+1)\,\mu^{i} L^{(i+1)}$$

$$I_1 \equiv -\frac{\partial \mathcal{L}(\mu)}{\partial(\bar{\partial}\phi)} \frac{\partial \mathcal{L}(\mu)}{\partial(\partial\phi)}\, \partial\phi\, \bar{\partial}\phi = -\sum_{i,j} \mu^{i+j}\, \tau^{(i)} \bar{\tau}^{(j)}$$

$$I_2 \equiv \frac{1}{4}\left[\frac{\partial \mathcal{L}(\mu)}{\partial(\partial\phi)}\, \partial\phi + \frac{\partial \mathcal{L}(\mu)}{\partial(\bar{\partial}\phi)}\, \bar{\partial}\phi - 2\,\mathcal{L}(\mu) \right] = \sum_{i,j} \mu^{i+j}\, \omega^{(i)} \omega^{(j)}$$

where $\tau^{(i)}$, $\bar{\tau}^{(i)}$, $\omega^{(i)}$ are the components of the stress tensor for the $L^{(i)}$ term in the Lagrangian, namely

$$\tau^{(i)} = -\frac{\partial L^{(i)}}{\partial(\bar{\partial}\phi)}\, \partial\phi\,, \qquad \bar{\tau}^{(i)} = -\frac{\partial L^{(i)}}{\partial(\partial\phi)}\, \bar{\partial}\phi\,, \qquad \omega^{(i)} = \frac{1}{2}\left[\frac{\partial L^{(i)}}{\partial(\partial\phi)}\, \partial\phi + \frac{\partial L^{(i)}}{\partial(\bar{\partial}\phi)}\, \bar{\partial}\phi - 2L^{(i)} \right]$$

Altogether we find that $L^{(i+1)}$ is determined entirely from $L^{(i)}$

$$L^{(i+1)} = - \frac{1}{i+1} \sum_{j=0}^{i} \left[\tau^{(i)} \tau^{(j-i)} - \omega^{(i)} \omega^{(j-i)} \right]$$

In this way we can determine the Lagrangian recursively to all orders.

Let us see how this works for the first few orders. Using $L^{(0)} = \partial\phi \, \bar\partial\phi$ we find

$$\tau^{(0)} = -(\partial\phi)^2 , \qquad \bar\tau^{(0)} = -(\bar\partial\phi)^2 , \qquad \omega^{(0)} = 0$$

hence the linear term in the Lagrangian $L^{(1)}$ reads

$$L^{(1)} = -\left[\tau^{(0)} \bar\tau^{(0)} - \omega^{(0)\,2} \right] = -(\partial\phi)^2 (\bar\partial\phi)^2$$

Now, from $L^{(1)}$ we can determine $\tau^{(0)}$, $\bar\tau^{(0)}$, and $\omega^{(0)}$ such that

$$\tau^{(1)} = 2(\partial\phi)^3 (\bar\partial\phi) , \qquad \bar\tau^{(1)} = 2(\partial\phi)(\bar\partial\phi)^3 , \qquad \omega^{(1)} = -(\partial\phi)^2(\bar\partial\phi)^2$$

Consequently, the quadratic term $L^{(2)}$ is given by

$$L^{(2)} = -\frac{1}{2}\left[\tau^{(0)} \bar\tau^{(1)} + \tau^{(1)} \bar\tau^{(0)} - \theta^{(0)}\theta^{(1)} - \theta^{(1)}\theta^{(0)} \right] = 2(\partial\phi)^3 (\bar\partial\phi)^3$$

In this way we can solve for the deformed Lagrangian $\mathcal{L}(\mu)$ to all orders

$$\mathcal{L}(\mu) = \sum_{i=0} \mu^i \mathcal{L}^{(i)}$$

$$= \partial\phi\bar{\partial}\phi - \mu\,(\partial\phi\bar{\partial}\phi)^2 + 2\mu^2(\partial\phi\bar{\partial}\phi)^3 + \mathcal{O}[\mu^3(\partial\phi\bar{\partial}\phi)^4]$$

$$= \sum_{i=0} \mu^i \binom{1/2}{i+1} 2^{2i+1} \left[\mathcal{L}^{(0)}\right]^{i+1}$$

$$= \frac{1}{2\mu}\left(\sqrt{1-4\mu\,\partial\phi\bar{\partial}\phi} - 1\right)$$

$$= -\frac{1}{2\mu} + \mathcal{L}_{NG}$$

where $\mathcal{L}_{NG}$ is the Nambu-Goto action of a string in a three-dimensional target space in the static gauge

$$\mathcal{L}_{NG} = \frac{1}{l_s^2}\sqrt{\det \partial_\alpha X^\mu \partial_\beta X_\mu}$$

where we identify

$$X^0 + X^1 = z, \qquad X^0 - X^1 = \bar{z}, \qquad X^3 = l_s^{1/2}\,\phi, \qquad l_s^2 = \mu.$$

Comments:

- If we consider D-2 free bosons at the IR fixed point, i.e. if $L^{(0)} = \sum_{i=1}^{D-2} \partial\phi_i \bar{\partial}\phi_i$, then the deformed classical action is the Nambu-Goto action of a D-dimensional string in the static gauge, namely

$$\mathcal{L}(\mu) = -\frac{1}{2\mu} + \mathcal{L}_{NG}, \qquad \mathcal{L}_{NG} = \frac{1}{2\mu}\sqrt{\det(\partial_\alpha X^\mu \partial_\beta X_\mu)} = \frac{1}{2\mu}\sqrt{1 + 4\mu L^{(0)} - 4\mu^2 B}$$

where $B = \Sigma_i (\partial\phi_i)^2 \Sigma_i (\bar{\partial}\phi_i)^2 - L^{(0)2}$.

- However, as discussed in the first few lectures, this action is incompatible with the lightcone spectrum used to derive the S-matrix (except for D=3, 26). We will soon see that this is the same spectrum of a $T\bar{T}$-deformed CFT at the quantum level.
- As a result, the action of a $T\bar{T}$-deformed QFT generically receives corrections from an infinite number of finite counterterms!

Alternatively, we can derive the deformed classical action in a covariant way where the stress tensor and the deformation are defined as

$$T_{\mu\nu}(\mu) = -\frac{2}{\sqrt{g}} \frac{\delta S(\mu)}{\delta g^{\mu\nu}} = g_{\mu\nu} \mathcal{L}(\mu) - 2 \frac{\partial \mathcal{L}(\mu)}{\partial g^{\mu\nu}}$$

$$(T\bar{T})_\mu = \frac{1}{2} \epsilon^{\mu\alpha} \epsilon^{\nu\beta} T_{\mu\nu} T_{\alpha\beta} = \mathcal{L}(\mu)^2 - 2 \mathcal{L}(\mu) g^{\mu\nu} \frac{\partial \mathcal{L}(\mu)}{\partial g^{\mu\nu}} + 2 \epsilon^{\mu\alpha} \epsilon^{\nu\beta} \frac{\partial \mathcal{L}(\mu)}{\partial g^{\mu\nu}} \frac{\partial \mathcal{L}(\mu)}{\partial g^{\alpha\beta}}$$

For example, for the free boson, the original Lagrangian reads

$$\mathcal{L}(0) = L^{(0)} = \frac{1}{2} g^{\mu\nu} X_{\mu\nu}, \qquad X_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi$$

Hence the deformed Lagrangian can only depend on $X_{\mu\nu}$ so that $T\bar{T}$ becomes

$$T\bar{T} = \mathcal{L}^2(\mu, x) - 2 \mathcal{L}(\mu, x) \, x \, \partial_x \mathcal{L}(\mu, x)$$

which leads to the differential equation

$$\partial_\mu \mathcal{L}(\mu, x) + (x \, \partial_x - 1) \mathcal{L}(\mu, x) = 0.$$

This is also known as inviscid Burger's equation, which can be made manifest by letting

$$\mathcal{L}(\mu, x) = \sqrt{x}\, f(\mu, t), \qquad y = -\frac{1}{\sqrt{x}}$$

whereupon the differential equation becomes

$$\partial_\mu f(\mu, y) + f(\mu, y)\, \partial_y f(\mu, y) = 0.$$

$f =$ velocity field

$\mu =$ time

The solution to this equation yields back the Nambu-Goto action.

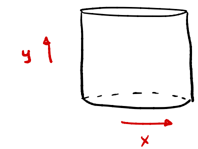
Using this approach we can also find the deformed action of a $T\bar{T}$-deformed free Dirac fermion which is given in terms of $\tilde{X}_{ab} = (i/2)\left(\bar{\Psi}\gamma_{(a}\nabla_{b)}\Psi - \nabla_{(a}\bar{\Psi}_{b)}\Psi\right)$ by

$$\mathcal{L}(\mu) = \mathcal{L}(0) + \frac{\mu}{4}\left[(\mathrm{Tr}\,\tilde{X})^2 - \mathrm{Tr}\,\tilde{X}^2\right], \qquad \mathcal{L}(0) = \frac{i}{2}\bar{\Psi}\gamma^\mu\nabla_\mu\Psi.$$

Due to the Grassmann nature of Ψ, the deformed Lagrangian is finite at the classical level and is not the appropriate description of the $T\bar{T}$-deformed theory. The appropriate description is via physical quantities like the s-matrix and the spectrum.

The spectrum of $T\bar{T}$-deformed QFTs

Let us consider an eingenstate $|n\rangle$ of the energy and momentum in the deformed theory, namely



$x \sim x + R$

$H \sim \partial_y$

$P \sim \partial_x$

$$H|n\rangle = E_n |n\rangle, \qquad P|n\rangle = P_n |n\rangle .$$

In addition we place the theory on a cylinder of size R. From the definition of the $T\bar{T}$ deformation we have

here we used the fact that $\langle n|T\bar{T}|n\rangle$ is constant

$$\partial_\mu E_n = \partial_\mu \langle n| H_{int} |n\rangle = -\int dx \; \partial_\mu \langle n| \mathcal{L}_{int} |n\rangle = -\frac{R}{\pi^2} \langle n|T\bar{T}|n\rangle$$

$H = H_0 + H_{int}$ $\qquad$ $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

Next we note that the components of the stress-energy tensor read (here $z = x + it$)

$$T_{zz} = \frac{1}{4}\left(T_{xx} - T_{tt} - 2i\,T_{xt}\right), \qquad T_{\bar{z}\bar{z}} = T_{zz}^{*}, \qquad T_{z\bar{z}} = \frac{1}{4}\left(T_{xx} + T_{tt}\right)$$

and their expectation values satisfy:

$$\langle n | T_{tt} | n \rangle = -\frac{E_n}{R}, \qquad \langle n | T_{xx} | n \rangle = -\frac{\partial E_n}{\partial R}, \qquad \langle n | T_{xt} | n \rangle = \frac{iP}{R}.$$

As a result, the expectation value of the $T\bar{T}$ operator reads:

$$\begin{aligned}
\langle n | T\bar{T} | n \rangle &= \langle n | T | n \rangle \langle n | \bar{T} | n \rangle - \langle n | \Theta | n \rangle^2 \\
&= 4\pi^2 \left(\langle n | T_{zz} | n \rangle \langle n | T_{\bar{z}\bar{z}} | n \rangle - \langle n | T_{z\bar{z}} | n \rangle^2 \right) \\
&= -\frac{\pi^2}{R} \left(E_n \frac{\partial E_n}{\partial R} + \frac{P_n^2}{R} \right)
\end{aligned}$$

Altogether, the deformed energy satisfies the differential equation

$$\boxed{\partial_\mu E_n = E_n \frac{\partial E_n}{\partial R} + \frac{P_n^2}{R}}$$

This equation is the inviscid Borger's equation with an additional driving force (the P_n^2 / R) term.

Let us now solve for the spectrum of the deformed theory. First, note that before the deformation, the momentum is quantized in units of R^{-1}

$$P(0) = \frac{m}{R}, \qquad m \in \mathbb{Z}.$$

Due to the quantization of the momentum, the latter cannot change under an infinitesimal deformation. As a result, the deformed momentum satisfies

$$P_n(\mu) = P_n(0).$$

We now note that the diffential equation for the deformed energy can be easily solved by turning it into a quadratic equation. This follows from the fact that RE_n is dimensionless and can only depend on dimensionless variables

$$R E_n(\mu) = e(\lambda), \qquad R P_n(\mu) = p(\lambda), \qquad \lambda = \frac{\mu}{R^2}.$$

We then have:

$$\partial_\mu \mathcal{E}_n = \frac{1}{R}\,\partial_\mu (R\,\mathcal{E}_n) = \frac{1}{R^3}\,\partial_\lambda e$$

$$\partial_R \mathcal{E}_n = \partial_R (R^{-1} e) = -\frac{1}{R^2}\,(e + 2\lambda\,\partial_\lambda e)$$

Consequently, the differential equation can be recast into

$$\frac{1}{R^3}\,\partial_\lambda (e + \lambda e^2 - \lambda p^2) = 0$$

$$\Rightarrow \quad e(0) = e(\lambda) + \lambda\, e(\lambda)^2 - \lambda p^2$$

Using $e = R\,\mathcal{E}_n$ and $p = R\,P_n$ we find

$$\boxed{\mathcal{E}_n(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu\,\mathcal{E}_n(0)}{R} + \frac{4\mu^2\,P_n(0)^2}{R^2}}\right)}$$

<u>Comments</u>:

- This formula is universal for any $T\bar{T}$-deformed QFT.* In particular, for a CFT with central charge c we have

* with Poincaré invariance when $\mu=0$.

$$E_n(0) = \frac{1}{R}\left(h_n + \bar{h}_n - \frac{c}{12}\right), \qquad P_n(0) = \frac{1}{R}\left(h_n - \bar{h}_n\right)$$

where h_n and $\bar{h}_n$ are the conformal weights of $|n\rangle$ in the undeformed theory,

$$L_0|n\rangle = \left(h_n - \frac{c}{24}\right)|n\rangle, \qquad \bar{L}_0|n\rangle = \left(\bar{h}_n - \frac{c}{24}\right)|n\rangle, \qquad (H = L_0 + \bar{L}_0, \quad P = L_0 - \bar{L}_0)$$

Consequently, the $T\bar{T}$ spectrum becomes

$$E_n(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu}{R}\left(h_n + \bar{h}_n - \frac{c}{12}\right) + \frac{4\mu^2}{R^2}(h_n - \bar{h}_n)^2}\right).$$

- If we identify

$$(h_n, \bar{h}_n) = (N, \tilde{N}), \qquad c = D-2, \qquad \mu = \ell_s^2,$$

then (up to a constant) we recover the lightcone spectrum of "the simplest theory of quantum gravity" introduced in lecture 2. Consequently the S-matrix of $T\bar{T}$-deformed CFTs is given by $S = \mathbb{1}\, e^{is\mu/4}$.

- Thus, we can think of $T\bar{T}$-deformed QFTs as toy models of quantum gravity that feature a non trivial S-matrix defined all the way to the UV, time delay in scattering amplitudes, and a minimum distance given by $\sqrt{\mu}$.
- When $\mu > 0$, the spectrum is well-behaved at all energies provided that μ satisfies

$$\mu \geqslant -\frac{R}{4E_0(0)}$$

where $E_0(0)$ is the energy of the ground state which for CFTs is negative, i.e. $E_0(0) = -\frac{c}{12}$.

- On the other hand, when $\mu < 0$, the spectrum becomes complex at all energies where

$$E_n(0) > \frac{R}{4|\mu|} + \frac{|\mu|}{R} P_n(0)^2 .$$

This feature of $T\bar{T}$-deformed QFTs has a geometrical counterpart in the context of holography where it corresponds either to the existence of a cutoff or the emergence of CTCs.

Plot of the spectrum ($P(0)=0$):