Aspects of holography and irrelevant deformations

Lecture 9 :

$T\bar{T}$ and Jackiw - Teitelboim gravity

References :

1706.06604

02.06.2021

In the previous lecture we derived the spectrum of general $T\bar{T}$-deformed QFTs.

In particular, we found that

- When the undeformed QFT consists of $D-2$ free scalars, the deformed spectrum is the same lightcone spectrum studied in the first half of the course

$$E(\mu) = -\frac{R}{2\mu}\left(1 - \sqrt{1 + \frac{4\mu}{R}\left(N + \tilde{N} - \frac{D-2}{12}\right) + \frac{4\mu^2}{R^2}(N-\tilde{N})^2}\right)$$

- Consequently, the S-matrix of this theory is simply given by

$$S = \mathbb{1}\, e^{is\mu/4}$$

- This S-matrix has many of the features of an (integrable) theory of 2d gravity including a minimum length $\sqrt{\mu}$, nonlocality, and a universal time delay $\Delta t \propto \mu E$ in scattering amplitudes.

- Crucially, we saw that this S-matrix is well defined in the UV, which allow us to circumvent the subtleties associated with irrelevant deformations. In particular, we argued that the appropriate description of $T\bar{T}$-deformed QFTs is via their $\underbrace{\text{spectrum or S-matrix}}_{\text{rigorous definition of } T\bar{T}}$ (instead of the action).

- In this lecture we will describe a more general and universal connection between $T\bar{T}$-deformed QFTs and theories of 2d gravity. We will see that

1. Turning on the $T\bar{T}$ deformation of a QFT is equivalent to coupling the QFT to flat space Jackiw-Teitelboim gravity.
2. The S-matrix of the $T\bar{T}$-deformed QFT receives a gravitational dressing factor that generalizes the $e^{i s \mu/4}$ term discussed previously.
3. This S-matrix can be obtained as a flat space limit of near AdS_2 holography.

Jackiw-Teitelboim (JT) gravity

In 2d the Einstein-Hilbert action is topological,

$$\int d^2\sigma \sqrt{g}\, R = \chi \qquad (\chi \equiv \text{Euler characteristic})$$

In order to have non-trivial dynamics we need to add degrees of freedom, e.g. a dilaton ϕ. The action of flat space Jackiw-Teitelboim gravity we are interested in is described by

$$S_{JT} = \int_\Sigma d^2\sigma \sqrt{g}\,(\phi R - \underbrace{\Lambda}_{\text{vacuum energy}}) - \underbrace{\Lambda \overset{\text{free parameter}}{R_0} \oint_{\partial\Sigma} du \sqrt{g_{uu}}}_{\text{boundary counterterm}}$$

induced metric at the boundary

The kinetic term is motivated by the fact that the dimensional reduction of the Einstein-Hilbert action in higher dimensions produces this term.

This action is closely related to the action of JT gravity used in near AdS_2

holography, which is given by

$$\tilde{S}_{JT} = \int_\Sigma d^2\sigma \sqrt{|g|}\left[\phi\left(R + \frac{2}{L^2}\right) + \overset{\Lambda = 0}{\Lambda}\right] + 2\,\phi_b \oint_{\partial\Sigma} du \sqrt{g_{uu}}\, K + 2\, c_0 \oint_{\partial\Sigma} du \sqrt{g_{uu}}$$

scale of AdS_2 — boundary value of the dilaton — trace of the extrinsic curvature

The two actions are related by $\phi \to \phi + \phi_0$, which shifts Λ and ϕ_b by:

$$\Lambda \to \Lambda + \frac{2\phi_0}{L^2}, \qquad \phi_b \to \phi_b + \phi_0$$

Hence, the physical combination of parameters is

$$\bar{\Lambda} = \Lambda - \frac{2}{L^2}\phi_b$$

In near AdS_2 holography one sets $\Lambda = 0$. For the flat space limit $L \to \infty$ of JT gravity we are interested in we must choose $\phi_b = 0$ instead. Note that after taking the $L \to \infty$ limit it's no longer possible to shift Λ.

Let us now consider (flat space) JT gravity coupled to matter fields ψ_i

$$S = S_{JT}[\phi, g] + \underbrace{S_{matter}[\psi_i, g]}_{\text{no coupling to } \phi}$$

The EOM read

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{|g|}} \frac{\delta S_{matter}}{\delta g^{\mu\nu}}$$

$$g_{\mu\nu}: \quad (\partial_\mu \partial_\nu - g_{\mu\nu} \Box)\phi - \frac{\Lambda}{2} g_{\mu\nu} = \frac{1}{2} T_{\mu\nu}$$

$$\phi: \quad R = 0$$

We see that the EOM of the dilaton sets all metric solutions to be flat. However the full gravitational dynamics of the theory also includes the dilaton which is non trivial.

Our goal will be to derive the contribution of gravity to the S-matrix of matter scattering on the Minkowski vacuum. The latter can be obtained by setting $T_{\mu\nu} = 0$

and is given in lightcone coordinates $\sigma^{\pm} \equiv \frac{\sigma^0 \pm \sigma^1}{\sqrt{2}}$ by

integration constant

$$g_{\mu\nu} = \eta_{\mu\nu} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \qquad \phi = -\frac{\Lambda}{4}\eta_{\mu\nu}\sigma^\mu\sigma^\nu + c$$

$$\phi = \underbrace{-\frac{\Lambda}{2}\sigma^+\sigma^-}_{\text{asymptotically nontrivial}} + c$$

Note that the vacuum is not invariant under translations, so it is not clear how it is possible to obtain a flat space (Poincaré-invariant) S-matrix via JT gravity. This puzzle is resolved by taking into account the additional symmetries of the theory. In order to see this it's convenient to use the conformal gauge $g_{\mu\nu} = e^{\Omega}\eta_{\mu\nu}$. The JT action is then given by

$$S_{JT} = \int d\sigma^+ d\sigma^- \left(4\phi\,\partial_+\partial_-\Omega - \Lambda e^{2\Omega}\right) + \dots$$

which is invariant under $\phi \to \phi + f_+(\sigma^+) + f_-(\sigma^-)$.

Thus, the vacuum is invariant under the combinated transformations

$$\sigma^{\pm} \to \sigma^{\pm} + a^{\pm}, \qquad \phi \to \phi + \frac{\Lambda}{2} a^{-}\sigma^{+} + \frac{\Lambda}{2} a^{+}\sigma^{-}.$$

The effect of JT gravity on the scattering problem is to provide a dynamical set of coordinates such that $\sigma^{\pm} \to x^{\pm}(\sigma^{\pm})$. This is similar to what happens for the critical string in the Polyakov formalism:

$$\underbrace{x^{0}, \quad x^{1}}, \quad x^{i} \quad \text{where } i = 2, \dots 25$$

think of these as dynamical coordinates for the string (which in the static gauge become $(x^0, x^1) = (\sigma^0, \sigma^1)$).

(in the static gauge the string transforms non linearly under Poincaré)

In the conformal gauge the EOM become

$$\partial_+^2 \phi = -\frac{1}{2} T_{++}, \qquad \partial_-^2 \phi = -\frac{1}{2} T_{--}$$

$$\partial_+ \partial_- \phi = \frac{1}{2}\left(\Lambda e^{\Omega} + T_{+-} \right) \qquad \partial_+ \partial_- \Omega = 0 .$$

Requiring all solutions to reduce to the vacuum at infinity implies that $\Omega = 0$ everywhere (i.e. all of the dynamics are in the dilaton).

We can then introduce the dynamical coordinates

$$\underbrace{x^{\pm} = \frac{2}{\Lambda} \partial_{\mp} \phi}_{x^{\pm} \to x^{\pm} + a^{\pm} \text{ under Poincaré}} \equiv \sigma^{\pm} + Y^{\pm}$$

note $Y^{\pm} \to 0$ at the boundary

In terms of these variables the EOM become:

$$\partial_+ Y^- = - \frac{T_{++}}{\Lambda}, \qquad \partial_- Y^+ = - \frac{T_{--}}{\Lambda}, \qquad \partial_+ Y^+ = \partial_- Y^- = \frac{T_{+-}}{\Lambda}$$

In particular we see that $\sigma^{\pm}$ and $Y^{\pm}$ are related by a non local change of coordinates. Integrating the first two equations

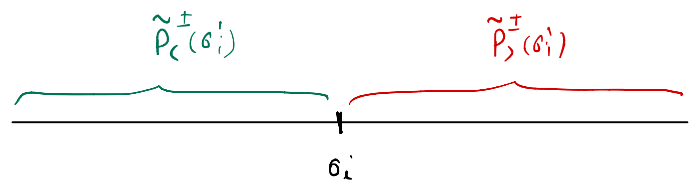
$$Y^{\pm}(\sigma^{\mp} = \infty) - Y^{\pm}(\sigma^{\mp} = -\infty) = \mp \frac{1}{2\Lambda} P^{\pm},$$

where $P^{\pm} \equiv \int_{-\infty}^{\infty} d\sigma^{\mp}\, T_{\mp\mp}$ is the total momentum along $\sigma^{\pm}$. We can then choose the integration constant such that

$$X^{+}(\sigma^{0}=-\infty, \sigma^{1}_{i}) = \frac{1}{2\Lambda}\left[\tilde{P}^{+}_{>}(\sigma^{1}_{i}) - \tilde{P}^{+}_{<}(\sigma^{1}_{i}) \right]$$

$$X^{-}(\sigma^{0}=-\infty, \sigma^{1}_{i}) = \frac{1}{2\Lambda}\left[\tilde{P}^{-}_{<}(\sigma^{1}_{i}) - \tilde{P}^{-}_{>}(\sigma^{1}_{i}) \right]$$

where $\tilde{P}^{\pm}_{>}(\sigma^{1}_{i})$ denotes the momenta of all the particles with $\underbrace{\sigma^{1} > \sigma^{1}_{i}}$ while $\tilde{P}^{\pm}_{<}(\sigma^{1}_{i})$ are the momenta of particles with $\sigma^{1} < \sigma^{1}_{i}$ ($\beta < \beta_{i}$). equivalent to $\beta > \beta_{i}$



$p^{0}_{i} = m \cosh \beta_{i}$

$p^{1}_{i} = m \sinh \beta_{i}$

Note: the choice of integration constants leading to $X^{\pm}(\sigma^{0}=-\infty, \sigma^{1})$ above does not seem to be unique and the dressed S-matrix obtained below depends critically on this choice.