Single-trace $T\bar{T}$ deformations and string theory

Lecture II

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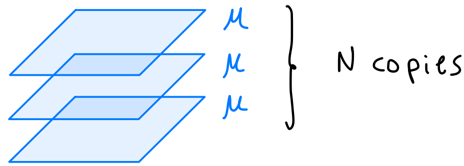
Symmetric orbifolds

$$\mathrm{Sym}^N \mathcal{M} \equiv \frac{\mathcal{M}^N}{S_N}$$

$\mathcal{M}$ → seed theory

S_N → symmetric group



- If $\mathcal{M}$ is a CFT: (1) $\mathrm{Sym}^N \mathcal{M}$ is also a CFT (modular inv.)

holographically desirable:

(2) $c = N c_{\mathcal{M}}$, $N \to \infty$ => large-c limit

(3) low energy spectrum doesn't grow with N

(4) correlators factorize at large-N

undesirable ← (5) infinite # of higher spin currents

(1) <u>Untwisted sector</u>: symmetrized product of states in $\mathcal{M}^N$

$$\Phi = \mathrm{Sym}\left(\otimes_{i=1}^{N} \phi^{(i)}\right) = \mathrm{Sym}\left(\phi^{(1)} \otimes \phi^{(2)} \otimes \dots \phi^{(N)}\right)$$

↓

large reduction on the # of states

- Single-particle states :

$$\Phi = \phi^{(1)} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1} + \mathbb{1} \otimes \phi^{(2)} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1} + \dots, \quad h_\Phi = h^{(1)}$$

e.g. the stress tensor

$$T = T^{(1)} \otimes \mathbb{1} \otimes \dots \otimes \mathbb{1} + \dots \equiv \sum_{i=1}^{N} T^{(i)}$$

$$T(z)\,T(0) = \sum_{i=1}^{N} T^{(i)}(z)\,T^{(i)}(0) = \frac{N c_{\mathcal{M}}/2}{z^4} + \frac{2T(0)}{z^2} + \frac{\partial T(0)}{z}$$

- Multi-particle states:

$$\Phi = \phi^{(1)} \otimes \phi^{(2)} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \phi^{(1)} \otimes \mathbb{1} \otimes \phi^{(3)} \otimes \cdots \otimes \mathbb{1} + \cdots , \qquad h_\Phi = \sum h^{(i)}$$

e.g. for the stress tensor

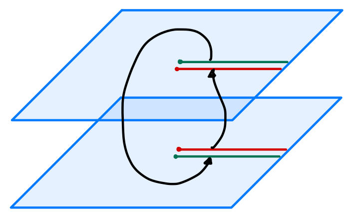
$$T^{(1)} \otimes T^{(2)} \otimes \mathbb{1} \otimes \cdots \otimes \mathbb{1} + \cdots = \sum_{i \neq j}^{N} T^{(i)} T^{(j)}$$

<u>Exercise</u>: show that $W_4 = \sum_{i=1}^{N} \left[(T^{(i)} T^{(i)}) - \frac{3}{10} \partial^2 T^{(i)} \right] - \alpha \sum_{i \neq j}^{N} T^{(i)} T^{(j)}$

is a spin-4 current, where $(AB)(z) = \oint \frac{dw}{2\pi i} A(w) B(z)$, $\alpha = \dfrac{\frac{22}{5 c_M} + 1}{N - 1}$

(2) <u>Twisted states</u>: states needed for modular invariance

$\phi_{(n)}$: "connects" n copies of M^N
$\downarrow$
twist $\qquad \mathcal{O}^{(j)}(e^{2\pi i} z) = \mathcal{O}^{g(j)}, \qquad g \in S_N$



- single-particle spectrum

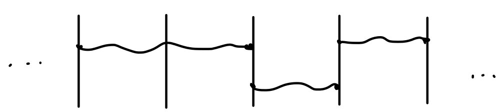
$$h_{(n)} = \frac{h}{n} + \frac{c_M}{24}\left(n - \frac{1}{n}\right), \qquad \text{for each } h \in M$$

$$h_{(n)} - \bar{h}_{(n)} \in n\mathbb{Z}$$

$$\boxed{E_L^{(n)} = h_{(n)} - \frac{n\, c_M}{24} = \frac{E_L}{n}, \qquad \text{for each } E_L \in M}$$

- multi-particle spectrum

$$E_L^{(1)} = \frac{E_L}{2}$$



$$\phi_{(2)} \otimes \phi^{(3)} \otimes \phi^{(4)}$$

Single - trace $T\bar{T}$

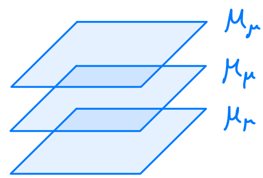
- A $\mathrm{Sym}^N \mathcal{M}$ allows to define the single trace $T\bar{T}$ deformation:

$$\frac{\partial S}{\partial \mu} = -4 \int (T\bar{T})_{ST}, \qquad (T\bar{T})_{ST} = \sum_{i=1}^{N} \left(T_{++}^{(i)} T_{--}^{(i)} - T_{+-}^{(i)\,2} \right)$$

$$\neq$$

$$T\bar{T} = \sum_{i,j}^{N} \left(T_{++}^{(i)} T_{--}^{(j)} - T_{++}^{(i)} T_{--}^{(j)} \right)$$

- $(T\bar{T})_{ST}$ takes $\mathrm{Sym}^N \mathcal{M} \longrightarrow \mathrm{Sym}^N \mathcal{M}_\mu$



- $\mathrm{Sym}^N \mathcal{M}_\mu$ has the same symmetries as $\mathcal{M}_\mu$ (modular invariance)

$$\Rightarrow \quad Z_N(\gamma\tau, \gamma\bar{\tau}; \gamma\hat{\mu}) = Z_N(\tau, \bar{\tau}; \hat{\mu})$$

The partition function:

$$Z_N(\tau, \bar{\tau}, \hat{\mu}) = \underbrace{Z_N^{\text{untwisted}}(\tau, \bar{\tau}; \hat{\mu})}_{\substack{\text{universal for} \\ \text{any } \mathrm{Sym}^N \mathcal{M}}} + \underbrace{Z_N^{\text{twisted}}(\tau, \bar{\tau}; \hat{\mu})}_{\substack{\text{depends on } \mathcal{M}\text{, determined} \\ \text{from modular invariance}}}$$

(1) Untwisted sector : (set $N=3$ for simplicity)

$$Z_N^{\text{untwisted}}(\tau, \bar{\tau}; \hat{\mu}) = \mathrm{Tr}_{\text{untwisted}} \left(q^{E_L(\hat{\mu})} \bar{q}^{E_R(\hat{\mu})} \right) = \sum_{i=1}^{3} Z^{(i)}(\tau, \bar{\tau}; \hat{\mu})$$

i) $\Phi_{(i)} = \phi_i^{(1)} \otimes \phi_i^{(2)} \otimes \phi_i^{(3)}$, e.g. the vacuum

$$E = 3E_i \quad \Rightarrow \quad Z^{(3)}(\tau, \bar{\tau}; \hat{\mu}) = Z(3\tau, 3\bar{\tau}; \hat{\mu}) \rightarrow$$

$Z(\tau, \bar{\tau}; \hat{\mu})$ is the partition function of the seed

ii) $\Phi_{(i,j)} = \phi_i^{(1)} \otimes \phi_i^{(2)} \otimes \phi_j^{(3)}$, $i \neq j$, e.g. stress tensor

$$\epsilon = 2\epsilon_i + \epsilon_j \quad \Rightarrow \quad z^{(2)}(c,\bar{c};\hat{\mu}) = \underbrace{z(2c,2\bar{c};\hat{\mu})\, z(c,\bar{c};\hat{\mu})}_{\supset\, q^{2\epsilon_i}\, q^{\epsilon_i}} - z(3c,3\bar{c};\hat{\mu})$$

iii) $\Phi_{(i,j,k)} = \phi_i^{(1)} \otimes \phi_j^{(2)} \otimes \phi_k^{(3)}$, $i \neq j \neq k$, $\epsilon = \epsilon_i + \epsilon_j + \epsilon_k$

$$z^{(1)}(c,\bar{c};\hat{\mu}) = \frac{1}{3!}\left[z(c,\bar{c};\hat{\mu})^3 - 3\, z^{(2)}(c,\bar{c},\hat{\mu}) - z^{(3)}(c,\bar{c};\hat{\mu}) \right]$$

Altogether, we have:

$$z_N^{\text{untwisted}} = \frac{1}{3!}\left[z(c,\bar{c};\hat{\mu})^3 + 3\, z(2c,2\bar{c};\hat{\mu})\, z(c,\bar{c};\hat{\mu}) + 2\, z(3c,3\bar{c};\hat{\mu}) \right]$$

<u>Structure of S_3:</u>

	conjugacy class		cycle		partition function
abc → abc	$\mathbb{1}^3$	$\{3,0,0\}$	$\mathbb{1}$	⟷	$z(c,\bar{c};\hat{\mu})$
→ bac, cba, acb	$\mathbb{Z}_2 \cdot \mathbb{1}$	$\{1,2,0\}$	$\mathbb{Z}_2$	⟷	$z(2c,2\bar{c};\hat{\mu})$
→ bca, cab	$\mathbb{Z}_3$	$\{0,0,1\}$	$\mathbb{Z}_3$	⟷	$z(3c,3\bar{c};\hat{\mu})$

$$\Rightarrow z_3^{\text{untwisted}} = \frac{1}{3!}\left(\text{cycle index of } S^3\right)$$

For any $\mathrm{Sym}^N \mathcal{M}$ we have

$$z_N^{\text{untwisted}} = \frac{1}{N!}\left(\text{cycle index of } S^N\right)$$

$$= \frac{1}{N!} \underbrace{\sum_{\{k_1,\dots,k_N\}}}_{\text{conjugacy class}} \underbrace{\frac{N!}{\prod_{n=1}^N n^{k_n}\, k_n!}}_{\#\text{ elements in conjugacy class}} \prod_{n=1}^{N} z(nc,n\bar{c};\hat{\mu})^{k_n}$$

$$\sum_{n=1}^{N} n k_n = N$$

Exercise: check this formula for $N=4$

(2) Twisted sector:

Note that $Z_N^{\text{untwisted}}$ is not modular invariant since for each $n>1$

$$Z(n\tau, n\bar\tau; \hat\mu) \xrightarrow[T]{} Z(n\tau, n\bar\tau; \hat\mu) \quad \text{since} \quad E_L(\hat\mu) - E_R(\hat\mu) = P(0)$$

$$Z(n\tau, n\bar\tau; \hat\mu) \xrightarrow[S]{} Z\left(-\frac{n}{\tau}, -\frac{n}{\bar\tau}; \frac{\hat\mu}{|\tau|^2}\right) = Z\left(\frac{\tau}{n}, \frac{\bar\tau}{n}; \frac{\hat\mu}{n^2}\right)$$

Make each $Z(n\tau, n\bar\tau; \hat\mu)$ invariant by adding its modular images.

For n prime

$$\underset{\circlearrowleft\, T}{Z(n\tau, n\bar\tau; \hat\mu)} + Z\left(\frac{\tau}{n}, \frac{\bar\tau}{n}; \frac{\hat\mu}{n^2}\right) + \overset{S\,\circlearrowleft}{\underset{T\,\circlearrowleft}{\sum_{\alpha=1}^{n-1} Z\left(\frac{\tau+\alpha}{n}, \frac{\bar\tau+\alpha}{n}; \frac{\hat\mu}{n^2}\right)}}$$

S ; $T^\alpha,\ \alpha < N$

Exercise: show that $\sum_{\alpha=1}^{n-1} Z\left(\frac{\tau+\alpha}{n}, \frac{\bar\tau+\alpha}{n}; \frac{\hat\mu}{n}\right)$ is invariant under S transformations when n is a prime.

For general n, the modular invariant combination is

$$T_n \cdot Z(\tau, \bar\tau; \hat\mu) = \sum_{\gamma | n} \sum_{\alpha=0}^{\gamma-1} Z\left(\frac{n\tau + \alpha\gamma}{\gamma^2}, \frac{n\bar\tau + \alpha\gamma}{\gamma^2}; \frac{\hat\mu}{\gamma^2}\right)$$

$\hookrightarrow$ generalization of the Hecke operator

$$T_n \cdot Z(\tau, \bar\tau; \hat\mu) = Z(n\tau, n\bar\tau; \hat\mu) + \underbrace{\sum_{\substack{\gamma | n \\ \gamma \neq 1}} \sum_{\alpha=0}^{\gamma-1} Z\left(\frac{n\tau + \alpha\gamma}{\gamma^2}, \frac{n\bar\tau + \alpha\gamma}{\gamma^2}; \frac{\hat\mu}{\gamma^2}\right)}_{\text{twisted sector}}$$

Partition function of $\mathrm{Sym}^N \mathcal{M}_\mu$:

$$Z(n\tau, n\bar{\tau}; \hat{\mu}) \rightarrow T_n \cdot Z(\tau, \bar{\tau}; \hat{\mu}), \qquad T_1 = \mathbb{1}$$

$$\boxed{Z_N(\tau, \bar{\tau}; \hat{\mu}) = \frac{1}{N!} \sum_{\{k_1, \dots k_N\}} \frac{N!}{\prod_{n=1}^N n^{k_n} \, k_n!} \prod_{n=1}^N \left[T_n \cdot Z(\tau, \bar{\tau}; \hat{\mu}) \right]^{k_n}}$$

Generating functional:

$$\mathcal{Z}(\tau, \bar{\tau}; \hat{\mu}) = \exp\left(\sum_{n=1}^{\infty} \frac{p^n}{n} T_n \cdot Z(\tau, \bar{\tau}; \hat{\mu}) \right) = \sum_{n=1}^{\infty} p^n Z_n(\tau, \bar{\tau}; \hat{\mu})$$

<u>Comments</u>:

- Same formulae as CFT with more general definition of T_n
- At large N (large c):

$$Z_N(\tau, \bar{\tau}; \hat{\mu}) \approx \begin{cases} e^{-\beta E_0(\hat{\mu})}, & \beta > 2\pi \\ e^{-\beta' E_0(\hat{\mu}')}, & \beta < 2\pi \end{cases}$$

where $E_0(\hat{\mu}) = \frac{N}{2\hat{\mu}} \left(\sqrt{1 - \frac{c\hat{\mu}}{3N}} - 1 \right) \rightarrow$ different from double trace:

$$E_0(\hat{\mu}) = \frac{1}{2\hat{\mu}} \left(\sqrt{1 - \frac{c\hat{\mu}}{3}} - 1 \right)$$

In the microcanonical ensemble:

$$\boxed{S(\hat{\mu}) = 2\pi \left(\sqrt{\frac{c}{6} R E_L(\mu) \left[1 + \frac{2\mu}{RN} E_R(\mu) \right]} + \sqrt{\frac{c}{6} R E_R(\mu) \left[1 + \frac{2\mu}{RN} E_L(\mu) \right]} \right)}$$

$\hookrightarrow$ differs slightly from the double trace case

The spectrum of twisted states

$$T_n \cdot Z(\tau,\bar\tau;\hat\mu) = \sum_{\gamma|n}\sum_{\alpha=0}^{\gamma-1} Z\left(\frac{n\tau+\alpha\gamma}{\gamma^2}, \frac{n\bar\tau+\alpha\gamma}{\gamma^2}; \frac{\hat\mu}{\gamma^2}\right)$$

$\hookrightarrow \gamma \neq 1 \Rightarrow$ twisted sector

In order to get the spectrum write:

$$Z\left(\frac{n\tau+\alpha\gamma}{\gamma^2}, \frac{n\bar\tau+\alpha\gamma}{\gamma^2}; \frac{\hat\mu}{\gamma^2}\right) = \mathrm{Tr}\left(q^{(n/\gamma^2)E_L(\hat\mu/\gamma^2)}\, \bar q^{(n/\gamma^2)E_R(\hat\mu/\gamma^2)}\, e^{(2\pi i\alpha/\gamma)J(0)}\right)$$

$$\sum_{\alpha=0}^{\gamma-1} e^{(2\pi i\alpha/\gamma)J(0)} = \gamma\,\delta_{J(0)\bmod\gamma} \rightarrow \text{this condition is already satisfied by } \mathrm{Sym}^N \mathcal{M}$$

$$\begin{aligned}\Rightarrow T_n\cdot Z(\tau,\bar\tau;\hat\mu) = &\sum_{E_{L,R}} \rho(E_L,E_R)\, q^{nE_L(\hat\mu)}\,\bar q^{nE_R(\hat\mu)}\\ &+ \sum_{\substack{\gamma|n\\ \gamma\neq\{1,n\}}}\sum_{E_{L,R}} \gamma\,\rho(E_L,E_R)\, q^{(n/\gamma^2)E_L(\hat\mu/\gamma^2)}\,\bar q^{(n/\gamma^2)E_R(\hat\mu/\gamma^2)}\,\delta_{J\ldots}\\ &+ \sum_{E_{L,R}} n\,\rho(E_L,E_R)\, q^{(1/n)E_L(\hat\mu/n^2)}\,\bar q^{(1/n)E_R(\hat\mu/n^2)}\,\delta_{J\bmod n}\end{aligned}$$

① untwisted multiparticle state

$$\phi^{(1)}\otimes\phi^{(2)}\otimes\ldots\otimes\phi^{(n)} \Rightarrow E_{L,R}^{\text{total}} = n\,E_{L,R}$$

② twisted multiparticle state (iff $n \neq$ prime) with twist $\gamma < n$

$$\underbrace{\phi_{(\gamma)}\otimes\phi_{(\gamma)}\otimes\ldots\otimes\phi_{(\gamma)}}_{n/\gamma} \Rightarrow E_{L,R}^{\text{total}} = \frac{n}{\gamma}\,E_{L,R}^{(\gamma)}$$

③ twisted single particle state

$$\mathrm{Sym}\left(\Phi_{(n)} \otimes \mathcal{H} \otimes \cdots \otimes \mathcal{H}\right) \Rightarrow \underbrace{\mathcal{E}^{(n)}_{L,R}(\hat{\mu}) = \frac{1}{n} \mathcal{E}_{L,R}(\hat{\mu}/n^2)}_{\substack{\text{same formula for CFT} \\ \text{up to the rescaling of } \hat{\mu}}}$$

The spectrum of single particle states in single-trace $T\bar{T}$-def CFTs can be written as ($n=1$ for the untwisted sector)

$$\boxed{E^{(n)}_{L,R}(0) = \mathcal{E}^{(n)}_{L,R}(\mu) + \frac{2\mu}{n} \mathcal{E}^{(n)}_{L}(\mu)\, \mathcal{E}^{(n)}_{R}(\mu), \qquad J^{(n)}(\mu) = J^{(n)}(0) \in \mathbb{Z}}$$

<u>Comments</u>

- Single-trace spectrum and density of states: additional factors of n and N. Single-trace spectrum contains sums of $\sqrt{}$'s!
- Modular invariance is compatible with both the single and double trace spectra.
- The entropy in the single-trace case is the same as Cardy's formula in $\mathrm{Sym}^N \mathcal{M}$ $\Rightarrow$ no change in the # of dofs.

$$S(\hat{\mu}) = 2\pi\left(\sqrt{\frac{c}{6} R\, \mathcal{E}_L(\mu)\left[1 + \frac{2\mu}{RN} \mathcal{E}_R(\mu)\right]} + L \leftrightarrow R\right)$$

$$= 2\pi\left(\sqrt{\frac{c}{6} R\, E^{(N)}_L(0)} + \sqrt{\frac{c}{6} R\, E^{(N)}_R(0)}\right)$$

$$= S(0)$$

↪ maximally-twisted state

String theory on AdS_3

Let us consider string theory on $AdS_3 \times S^3 \times T^4$. We focus on the bosonic AdS_3 sector with NS-NS flux. The low energy effective description is IIB SUGRA:

$$I = \frac{2\pi}{(2\pi l_s)^8} \int d^{10}x \sqrt{|g|}\, e^{-2\phi} \left\{ R - 4\, \partial_\mu \phi\, \partial^\mu \phi - \frac{1}{12} \underbrace{H_{\mu\nu\alpha} H^{\mu\nu\alpha}}_{H = dB} \right\}$$

The background features electric and magnetic charges

$$Q_e = \frac{1}{(2\pi l_s)^6} \int_{S^3 \times T^4} e^{-2\phi} * H = p, \qquad Q_m = \frac{1}{(2\pi l_s)^2} \int_{S^3} H = k$$

↳ F1 brane is electrically charged under B

↳ NS5 brane is magnetically charged under B

Related to the fact that $AdS_3 \times S^3 \times T^4$ originates from the near horizon limit of p F1 and k NS5 branes ⟹ S-dual of the D1-D5 system.

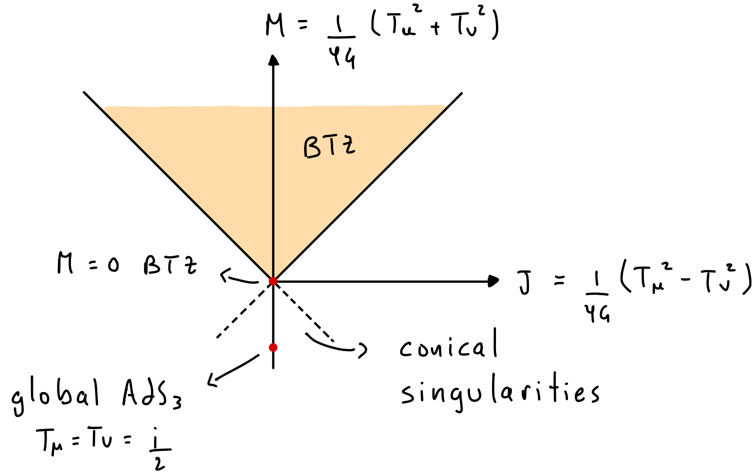
The space of asymptotically AdS_3 solutions can be parametrized as

$$ds^2 = \underbrace{l^2}_{\text{AdS scale}} \left\{ \frac{dr^2}{4(r^2 - 4T_u^2 T_v^2)} - r\, \underbrace{du\, dv}_{u = t+\varphi,\ v = t-\varphi,\ \varphi \sim \varphi + 2\pi} + T_u^2\, du^2 + T_v^2\, dv^2 \right\} + ds_{S^3} + ds_{T^4}$$

$$B = \frac{l^2}{2} r\, du \wedge dv + B_{\Omega_3}$$

$$e^{2\phi} = \frac{k}{p}, \qquad k \equiv \frac{\ell^2}{l_s^2}, \qquad k \gg 1, \quad p \gg 1 \longrightarrow \text{weak string coupling}$$

↳ semiclassical limit



String theory on AdS_3 with NS-NS flux admits a perturbative worldsheet description as an $SL(2,\mathbb{R}) \times SU(2)$ WZW model,

$$S_{WZW} = \frac{k}{16\pi} \int_{\partial\mathcal{M}} d^2z \; Tr(\partial_a g^{-1} \partial^a g) + k\,\Gamma, \qquad g \in SL(2,\mathbb{R})$$

$$\Gamma \propto \int_{\mathcal{M}} d^3x \, \epsilon^{abc} \, Tr(g^{-1}\partial_a g \partial_b g^{-1} \partial_c g)$$

or as a NLSM. The AdS_3 part of the action is:

$\epsilon^{z\bar{z}} = -1$ target space coords: (ρ, u, v)

$$S = -\frac{1}{2l_s^2} \int d^2z \left(\sqrt{-\eta}\, \eta^{ab} G_{\mu\nu} + \epsilon^{ab} B_{\mu\nu} \right) \partial_a X^\mu(z,\bar{z}) \, \partial_b X^\nu(z,\bar{z})$$

worldsheet coords $(z, \bar{z}) = (\tau+\sigma, \tau-\sigma)$

worldsheet metric $ds^2 = -dz\,d\bar{z}$

$$= \frac{1}{l_s^2} \int d^2z \; M_{\mu\nu} \, \partial X^\mu \bar{\partial} X^\nu, \qquad M_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu}$$

For global AdS_3:

$$S = k \int d^2 z \left(\frac{\partial r \bar{\partial} r}{4r^2 - 1} - \frac{1}{4} \partial u \bar{\partial} u - \frac{1}{4} \partial v \bar{\partial} v - r \bar{\partial} u \partial v \right)$$

<u>Exercise</u>: Parametrize the group elements of $SL(2,\mathbb{R})$ by $T^3 = -\frac{i}{2}\sigma^2$, $T^{\pm} = \frac{1}{2}(\sigma^3 \pm i\sigma^1)$. Use $g = e^{uT^3} e^{\frac{1}{2}\log(2r + \sqrt{4r^2-1})(T^+ + T^-)} e^{vT^3}$ to show the EOM of the NLSM match those of the WZW model ($T_u = T_v = \frac{i}{2}$)

The worldsheet action is invariant under

$$\hat{SL}(2,\mathbb{R})_L \times \hat{SL}(2,\mathbb{R})_R$$

↳ obtained in the WZW formulation by $\delta g = \omega(z) g + g \bar{\omega}(\bar{z})$

↳ or from the isometries of AdS_3 with wordsheet dependent parameters (e.g. $\delta u(z,\bar{z}) = \alpha(z)$)

These symmetries are generated by the chiral WZW currents

$$J^a(z) = -k \, \mathrm{Tr}(T^a \partial g g^{-1}), \qquad \bar{J}^a(\bar{z}) = k \, \mathrm{Tr}(T^a g^{-1} \bar{\partial} g)$$

which are related to the Noether currents of NLSM by

$$j_{\text{Noether}}(z,\bar{z}) = J^a(z) + \theta(z,\bar{z}) \longrightarrow \text{"topological" term,}$$

Example: for translations along x^μ ($\delta x^\mu = \xi^\mu$) we have

$$j_\xi = l_s^{-2} \xi^\nu (M_{\nu\mu} \bar{\partial} x^\mu \partial + M_{\mu\nu} \partial x^\mu \bar{\partial})$$

$$\xi = \partial_u \Rightarrow j_{\partial u} = \underbrace{\frac{k}{2}(\partial u - 2r \partial v)}_{J^3} \bar{\partial} + \frac{k}{4}(\bar{\partial} u \partial - \partial u \bar{\partial})$$

For the $M=0$ BTZ black hole all Noether currents are chiral, i.e.

$$j_{\text{Noether}} = J^a \bar{\partial} \qquad \text{or} \qquad j_{\text{Noether}} = \bar{J}^a \partial$$

From the chiral part of j_{Noether} or the WZW current we obtain:

$$J^a_n = \frac{1}{2\pi} \oint e^{inz} J^a \quad \Rightarrow \quad [J^3_n, J^3_m] = -\frac{k}{2} n \delta_{n+m}$$

$$[J^3_n, J^\pm_m] = \pm J^\pm_{n+m}$$

$$[J^+_n, J^-_m] = -2J^3_{n+m} + k n \delta_{n+m}$$

& particularly interested in J^3_0

Energy and angular momentum of the string $(\epsilon = \epsilon_L + \epsilon_R,\ P = \epsilon_L - \epsilon_R)$

$$\epsilon_L = \frac{1}{2\pi}\oint j_{\partial u} = \underbrace{\frac{1}{2\pi}\oint J^3}_{J^3_0} + \frac{k}{8\pi}\underbrace{\oint \partial_\sigma u}_{\text{winding}}, \qquad \epsilon_R = \frac{1}{2\pi}\oint j_{\partial v}$$

The worldsheet stress energy tensor (at large k):

$$T_{ab} = \frac{2}{\sqrt{-\eta}} \frac{\delta S}{\delta \eta^{ab}} \quad \Rightarrow \quad T = -l_s^{-2} G_{\mu\nu} \partial X^\mu \partial X^\nu$$

<u>Exercise</u>. compute the WZW currents and verify that the stress tensor matches the Sugawara stress tensor at large k

$$T = \frac{1}{k-2} J^a J^a = \frac{1}{k-2}\left[\frac{1}{2}(J^+J^- + J^-J^+) - (J^3)^2\right]$$

<u>Virasoro algebra</u>:

$$L_n = -\frac{1}{2\pi}\oint T e^{inz}, \qquad [L_n, L_m] = (n-m)L_{n+m} + \frac{\tilde{c}}{12} n(n^2-1)\delta_{n+m},$$

$$\tilde{c} = \frac{3k}{k+2}$$

The spectrum

Spectrum of strings on AdS_3 $\rightarrow$ unitary reps of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

$$|j, m\rangle \quad \longrightarrow \quad c_2 |j,m\rangle = -j(j-1)|j,m\rangle, \qquad J_0^3 |j,m\rangle = m |j,m\rangle$$

$$c_2 \equiv \sum_a J_0^a J_0^a, \qquad J_0^a = \frac{1}{2\pi}\oint J^a,$$

(1) Principal discrete reps (short strings)

$$D_j^{\pm} = \{ |j,m\rangle : \ m = \pm j, \pm(j+1), \pm(j+2), \dots \}, \qquad J_0^{\mp}|j, \pm j\rangle = 0$$

$$\text{unitarity}: \quad j \in \mathbb{R}_{>0}$$

$$\underbrace{\tfrac{1}{2}}_{} < j < \tfrac{k}{2}$$

wavefunction is $\mathcal{L}^2$ $\leftarrow$ $\frac{1}{2}$; $\frac{k}{2}$ $\rightarrow$ no ghost theorem

(2) Principal continous reps (long strings)

$$C_j^{\lambda} = \{ |j,m\rangle : \ m = \lambda, \lambda \pm 1, \lambda \pm 2, \dots \}, \qquad \lambda \in [0,1)$$

$$\text{unitarity}: \quad j = \tfrac{1}{2} + is, \quad s \in \mathbb{R}$$

Virasoro constraints:

$$(L_0 - 1)|\psi\rangle = 0 \quad \Rightarrow \quad \underbrace{-\frac{j(j-1)}{k-2}}_{\frac{c_2}{k-2}} + \underbrace{N}_{\text{level}} + h = 1$$

$\hookrightarrow$ internal manifold

$\hookrightarrow$ for $D_j^{\pm}$: N is bounded from above (finite string excitations)

$\hookrightarrow$ for C_j^{λ}: only solution has $N = h = 0$ (the tachyon)

This can be remedied by the inclusion of additional "flowed" representations.

<u>Spectral flow</u>

Let's consider the EOM of the NLSM with general T_u, T_v:

$$\delta S = -\frac{1}{\ell_s^2}\int d^2z \Big\{ (f(r) + \bar\partial u\, \partial v)\,\delta r - \left[\tfrac{1}{2}\partial\bar\partial u + \bar\partial(r\partial v)\right]\delta u$$
$$- \left[\tfrac{1}{2}\partial\bar\partial v + \partial(r\bar\partial u)\right]\delta v \Big\}$$

The EOM are invariant under the spectral flow transformation:

$$u \to u - wz\,, \qquad v \to v - w\bar z$$

In general (non AdS_3) spacetimes this is related to the existence of chiral currents.

<u>Exercise</u>: assume $M_{\mu\nu}$ is invariant under translations along X^m. Find the conditions on $M_{\mu\nu}$ such that $X^m \to X^m + wz$ leaves the EOM invariant.

We can use spectral flow to generate winding string solutions:

$$u(\tau),\ v(\tau),\ r(\tau) \xrightarrow{SF} \tilde u(\tau,\sigma),\ \tilde v(\tau,\sigma)$$

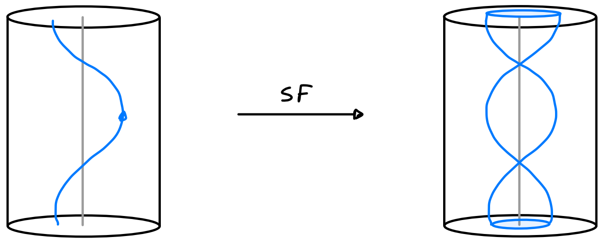
$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\text{EOM} \Rightarrow \text{geodesic eq.} \qquad \tilde\varphi = \frac{\tilde u - \tilde v}{2} = \frac{u-v}{2} - w\sigma$$

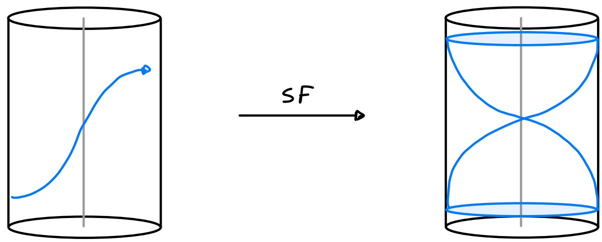
$$\Downarrow$$

$$\tilde\varphi \sim \tilde\varphi - 2\pi w$$

short strings:



long strings:



The spectral flow transformation induces a transformation of the currents that leaves the algebra unchanged, i.e. it's an automorphism of the $SL(2,\mathbb{R})$ algebra:

easy to check using the results of a previous exercise.

$$J^3 \to \tilde{J}^3 = J^3 - \frac{k}{2}\omega, \qquad J^\pm \to \tilde{J}^\pm = e^{\mp i z \omega} J^\pm, \qquad \omega \in \mathbb{Z}$$

When the group is non compact, spectral flow generates new reps.

Hence, for $SL(2,\mathbb{R})$ we have:

improved no-ghost theorem

$$D_j^{\pm,\omega}, \qquad C^{\lambda,\omega}_{\frac{1}{2}+is} \qquad \text{where} \qquad \frac{1}{2} < j < \frac{k-1}{2}$$

Flowed stress tensor:

$$T = \frac{1}{k}\left[\frac{1}{2}(J^+J^- + J^-J^+) - (J^3)^2\right] \Rightarrow \tilde{T} = T + \omega J^3 - \frac{k\omega^2}{4}$$

$$\Rightarrow \tilde{L}_0 = \frac{1}{2\pi}\oint \tilde{T} = L_0 + \omega J_0^3 - \frac{k\omega^2}{4}$$

The Virasoro constraints of a flowed state $|j,m\rangle \longrightarrow |\tilde{j},\tilde{m}\rangle$ become

$$(L_0 - 1)|\tilde{\psi}\rangle = 0 \quad \Rightarrow \quad -\frac{\tilde{j}(\tilde{j}-1)}{k} - w\tilde{J}_0^3 - \frac{kw^2}{4} + \underbrace{N + h}_{} = 1$$

- arbitrarily large
- w-flowed continuous reps.

Recall the left moving energy of the string is given by

$$E_L^{(w)} = \frac{1}{2\pi}\oint j_{\partial u} = \tilde{J}_0^3 + \frac{k}{4}w$$

The Virasoro constraints become

$$-\frac{\tilde{j}(\tilde{j}-1)}{k} - w E_L^{(w)} = 1 \quad \Rightarrow \quad E_L^{(1)} = -\frac{\tilde{j}(\tilde{j}-1)}{k} - 1$$

$$\Rightarrow \quad \underbrace{E_L^{(w)} = \frac{1}{w}E_L^{(1)}}_{}$$

energy of a twist-w state in a symmetric orbifold !

Comments :

- For $D_j^\pm$, $E_L^{(1)} = -\frac{\tilde{j}(\tilde{j}-1)}{k} - 1$ can not be generically solved since

$$\tilde{J}_0^3|\tilde{j},\tilde{m}\rangle = \tilde{m}|\tilde{j},\tilde{m}\rangle, \qquad \tilde{m} \in \mathbb{Z}$$

- For C_j^λ, we have $\tilde{m} = \lambda \pm n$ with $\lambda \in \mathbb{R}$, $n \in \mathbb{Z}$. Hence, it's possible to solve the Virasoro constraints !

  $\Rightarrow$ spectrum of long strings captured by a symmetric orbifold where w = twist of the state ($w=1 \rightarrow$ untwisted sector)

However, the holographic dual is not a $\underbrace{\text{symmetric orbifold}}$

in tension with the semiclassical limit $(k \gg 1)$ of sugra: no tower of higher spins fields

At $k=1$ we don't have a semiclassical approximation but:

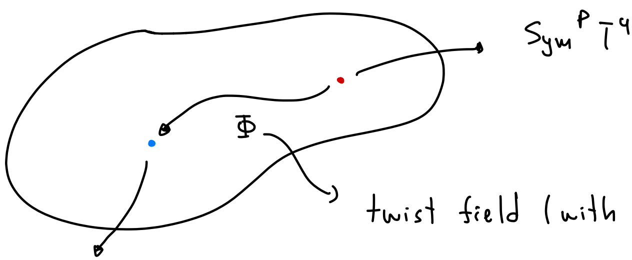
(1) there's an ∞ tower of massless higher spin states

(2) there are no discrete reps

(3) only the $\underbrace{s=0 \ C_j^{\lambda} \text{ reps}}_{\text{discrete spectrum}}$ are present

$\Rightarrow$ At $k=1$ the holographic dual is $\mathrm{Sym}^p T^4$ with $c=6p$

For general k we expect to move along the conformal manifold



twist field (with $n=2$) that partially destroys the symmetric product structure

CFT with $c = 6kp$

↳ long string spectrum still captured by a symmetric orbifold

↳ allows for the definition of single-trace deformations