Single-trace $T\bar{T}$ deformations and string theory

Lecture III

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The $T\bar{T}$ deformation in string theory

We come back to the question:

$$
\begin{tikzcd}
\text{UV:} & T\bar{T} \arrow[r, leftrightarrow] \arrow[d, squiggly] & ? \arrow[d, squiggly] \\
\text{IR:} & CFT_2 \arrow[r, leftrightarrow] & \text{string theory on } AdS_3
\end{tikzcd}
$$

At $k=1$ $\rightarrow$ single or double trace deformation

For any k $\rightarrow$ single trace deformation of the long string sector is more natural in string theory

Let us consider the $M=0$ BTZ black hole

$$ds^2_{AdS_3} = \ell^2 \left(d\rho^2 - e^{2\rho} du dv \right), \qquad 2r = e^{2\rho}$$

$$B_{AdS_3} = \frac{\ell^2}{2} e^{2\rho} du \wedge dv$$

The Noether currents are chiral and generate an $SL(2,\mathbb{R})$ algebra

$$\xi^- = \partial_u \qquad\qquad j^- = j_{\partial_u} = k e^{2\rho} \partial v \bar{\partial},$$

$$\xi^3 = u\partial_u - \frac{1}{2}\partial_\rho \qquad \Rightarrow \qquad j^3 = \ldots$$

$$\xi^+ = u^2 \partial_u - e^{-2\rho}\partial_v - u\partial_\rho \qquad\qquad j^+ = \ldots$$

<u>Exercise</u>: find the j^3 and j^+ Noether currents and confirm j^-, j^3, and j^+ are chirally conserved.

The $T\bar{T}$ operator from the worldsheet

Using the worldsheet currents we can construct a vertex operator that corresponds to the stress tensor in the dual CFT!

The ingredients we need are:

(1) the representation of the $SL(2,\mathbb{R})$ algebra in the dual CFT

$$J_0^- = -\partial_x\,, \qquad J_0^3 = -(x\partial_x + h)\,, \qquad J_0^+ = -(x^2\partial_x + 2hx)$$

auxiliary dual CFT coordinates

conformal weight in the dual CFT

(2) the primary fields

$$\Phi_h = \frac{1}{\pi}\left(\frac{1}{(u-x)(v-\bar{x})\,e^{\rho} - e^{-\rho}}\right)^{2h}$$

representation of both $SL(2,\mathbb{R})$ algebras, i.e.

$$[J_0^a, \Phi_h] = \mathcal{L}_{\xi^a}\Phi_h$$

↳ check: $C_2\,\Phi_h = \frac{1}{4}\Box\,\Phi_h \;\Rightarrow\; m^2 = -4C_2 = 4h(h-1)$

↳ dual CFT weight: (h,h), worldsheet: (Δ,Δ), $\Delta = \dfrac{-h(h-1)}{k-2}$

↳ $T \sim C_2 + \dots$

(3) The "soldering" current:

$$J(x;z) \equiv 2x\,j^3(z) - j^+(z) - x^2 j^-(z)\,, \qquad \partial_x^3 J(x;z) = 0$$

$$\frac{1}{2}\partial_x^2 J(x;z) = U^\dagger j^-(z)\,U\,, \qquad U = e^{xJ_0^-}\,, \qquad \frac{1}{2}\partial_x J = U^\dagger j^3 U\,, \qquad -J = U^\dagger j^+ U$$

↳ dual CFT weight: $(-1,0)$, worldsheet: $(1,0)$

Using the scaling dimensions of Φ_n, J, and $\bar J$ we can construct a vertex operator of weight $(2,0)$ in the dual CFT:

worldsheet: $(0,0)$, $(1,0)$, $(0,0)$, $(0,1)$

$$T(x) = \frac{1}{2K}\int d^2z \left(a_1\, J\, \partial_x^2 \Phi_1 + a_2\, \partial_x J\, \partial_x \Phi_1 + a_3\, \partial_x^2 J\, \Phi_1 \right) \bar J$$

dual CFT: $(2,0)$, $(-1,0)$, $(1,1)$, $(0,-1)$

The a_i can be fixed by imposing the physical state conditions or requiring T transforms as a tensor, such that

$$T(x) = \frac{1}{2K}\int d^2z \left(\partial_x J\, \partial_x \Phi_1 + 2\, \partial_x^2 J\, \Phi_1 \right) \bar J$$

$T(x)$ satisfies all of the desired properties for the stress tensor of the dual CFT, e.g. the $T(x)T(y)$ OPE.

Strategy to evaluate the OPE:

$$\partial_{\bar x}(\Phi_1 \bar J) = K\, \partial_{\bar z}\Phi_1 \quad \Rightarrow \quad \partial_{\bar x} T(x) = \frac{1}{2}\oint dz \left(\partial_x J\, \partial_x \Phi_1 + 2\, \partial_x^2 J\, \Phi_1 \right)$$

↳ vanishes within correlators up to contact terms

$$J_{x;z}\, J_{y;w} = K\frac{(y-x)^2}{(z-w)^2} + \frac{1}{z-w}\left[(y-x)^2\partial_y - 2(y-x) \right] J_{y;w}$$

$$\partial_{\bar x} T(x) T(y) = \frac{1}{4K}\int d^2w \oint dz \Big\{ \partial_x \underbrace{\overbrace{J(x;z)\, \partial_x \Phi_1(x;z)\, \partial_y J(y;w)}^{\text{single and double poles}}\, \partial_y \Phi_1(y;z)}_{\text{single pole}}\, \bar J(\bar y;\bar z) + \dots$$

$$\hookrightarrow \lim_{z\to w} \Phi_1(x;z)\, \Phi_1(y;w) = \underbrace{\delta^{(2)}(x-y)}\, \Phi_1(y;w)$$

$$\delta^{(2)}(x-y) = \frac{1}{\pi}\,\partial_{\bar x}\frac{1}{x-y}$$

Altogether we find

$$T(x)\,T(y) = \frac{3kp}{(x-y)^4} + \frac{2T(y)}{(x-y)^2} + \frac{\partial T(y)}{x-y}, \qquad c = 6kp \; ✓$$

$\partial_x^3 J(x;z) = 0 \Rightarrow$ no other chiral vertex operators with weight $h>2$ other than $\partial_x^{h-2} T(x)$.

There's another vertex operator we can construct with dimension $(2,2)$ in the dual CFT:

$$D(x,\bar{x}) = \int d^2z \, \underbrace{(\partial_x J \partial_x + 2\partial_x^2 J)}_{(1,0)} \underbrace{(\partial_{\bar{x}} \bar{J} \partial_{\bar{x}} + 2\partial_{\bar{x}}^2 \bar{J})}_{(0,1)} \underbrace{\Phi_1}_{(1,1)}$$

Using similar techniques we find

$$T(x)\,D(y,\bar{y}) = \frac{3k\,\bar{T}(\bar{y})}{(x-y)^4} + \frac{2D(y,\bar{y})}{(x-y)^2} + \frac{\partial_y D(y,\bar{y})}{x-y}, \qquad c_M = 6k$$

Let us consider the OPEs of T with the double and single trace versions of $T\bar{T}$ in a $\mathrm{Sym}^p M$ with $c_M = 6k$:

(1) $T\bar{T} \equiv \sum_{i,j=1}^{p} T^{(i)}\bar{T}^{(j)}$.

$$T(x)\,T\bar{T}(y) = \sum_{i,j,k} \underbrace{T^{(i)}T^{(j)}}_{\delta_{ij}} \bar{T}^{(k)} = \sum_{j,k} \frac{3k\,\bar{T}^{(k)}}{(x-y)^4} + 2\frac{T^{(j)}\bar{T}^{(k)}}{(x-y)^2} + \frac{\partial T^{(j)}\bar{T}^{(k)}}{x-y}$$

$$= \frac{3kp\,\bar{T}(y)}{(x-y)^4} + \frac{2T\bar{T}(y)}{(x-y)^2} + \frac{\partial T\bar{T}(y)}{x-y}$$

$$\frac{c}{2} = \frac{p\,c_M}{2}$$

(2) $(T\bar{T})_{ST} = \sum_{i=1}^{p} T^{(i)}\bar{T}^{(i)}$.

$$T(x)\, T\bar{T}(y) = \sum_{i,j} T^{(i)}\, T^{(j)} \bar{T}^{(j)} = \sum_j \frac{3k\bar{T}^{(j)}}{(x-y)^4} + 2\frac{T^{(j)}\bar{T}^{(j)}}{(x-y)^2} + \frac{\partial T^{(j)}\bar{T}^{(j)}}{x-y}$$

$$= \frac{3k\,\bar{T}(y)}{(x-y)^4} + \frac{2(T\bar{T})_{ST}}{(x-y)^2} + \frac{\partial(T\bar{T})_{ST}}{x-y}$$

c_M

$\Rightarrow$ $D(y,\bar{y})$ is the vertex operator for single trace $T\bar{T}$!

There are no other local vertex operators with weight $(2,2)$. In particular $T(x)\bar{T}(\bar{x}) = \int d^2z \left(\int d^2w \cdots \right)$ is nonlocal

Marginal worldsheet deformations

Consider deforming the worldsheet action by

$$\int d^2x\, (T\bar{T})_{ST} \sim \int d^2x\, D(x,\bar{x}) = \int d^2z \underbrace{\int d^2x\, \Phi_1}_{\text{constant}}\; \underbrace{\partial_x^2 J(x)}_{\downarrow\; 2 j_{\partial\mu}(z)}\; \underbrace{\partial_{\bar{x}}^2 \bar{J}(\bar{x})}_{\downarrow\; 2 j_{\partial\nu}(\bar{z})}$$

$$\Rightarrow \int d^2x\, (T\bar{T})_{ST} \sim \int d^2z\, j_{\partial\mu}(z)\, j_{\partial\nu}(\bar{z})$$

Comments.

- Translations along u ($= x$ at $\rho \to \infty$) are generated by

  $j^-(z)$ on the wordsheet

  $T(x)$ on the dual CFT

  $\Big\}$ precise sense in which $j_{\partial u}\, j_{\partial v} \sim T\bar{T}$

- The deformation is exactly marginal (on the worldsheet)

  $\hookrightarrow$ preserves shift symmetry (translations) along u, v

  $\hookrightarrow$ preserves conformal symmetry of the worldsheet

↳ generates another solution of SUGRA

- $D(x,\bar{x})$ is not known away from the CFT fixed point but we can work directly with the Noether currents j^- and $\bar{j}^-$.
- These currents are not chiral for other AdS_3 backgrounds

↳ must generalize the $\int d^2z\, j^-(z)\bar{j}^-(\bar{z})$ deformation

Let us rewrite the $T\bar{T}$ deformation in terms of the "currents" generating translations:

$$\dot{j}^{QFT}_{(x)} = T_{\mu x}\, dx^\mu, \qquad \dot{j}^{QFT}_{(\bar{x})} = T_{\mu\bar{x}}\, dx^\mu$$

$$d * j^{QFT}_{(x)} \propto \underbrace{(\partial_{\bar{x}} T_{xx} + \partial_x T_{\bar{x}x})}_{0} dx \wedge d\bar{x}, \qquad d * \dot{j}^{QFT}_{(\bar{x})} = 0$$

$$\partial_\mu S = -4\int d^2x\, (T_{xx}T_{\bar{x}\bar{x}} - T_{x\bar{x}}^2) = -4\int \underbrace{\dot{j}_{(x)} \wedge \dot{j}_{(\bar{x})}}$$

can be generalized to other defs, e.g $J\bar{T}$, $\tilde{j}_{(x)} = J_\mu dx^\mu$

The worldsheet deformation is proposed to be

$$\partial_{\hat{\mu}} S_{WS} = -4\int \dot{j}_{\partial\mu} \wedge \dot{j}_{\partial\nu},$$

$\hookrightarrow$ instantaneous

$$\ell\, \dot{j}_{(x)} = \dot{j}_{\partial\mu}, \qquad \mu = \ell^2 \hat{\mu}$$

The deformation is exactly marginal and can be written as a current-current deformation after a change of coordinates.

Using the definition of S_{ws} and j_ξ

$$S_{ws} = \ell_s^{-2} \int d^2 z \; \partial X^T M \bar{\partial} X \; ,$$

$$j_\xi = -\ell_s^{-2} \left(\xi^T M \bar{\partial} X \, d\bar{z} + \partial X^T M \xi \, dz \right)$$

we obtain

$$\Gamma^{(u,v)} = 2\, \delta_\alpha^{[u} \delta_\beta^{v]}$$

$$\partial_{\hat{\mu}} S_{ws} = -4 \int j_{\partial_\mu} \wedge j_{\partial_v} \;\Rightarrow\; \partial_{\hat{\mu}} M = -\ell_s^{-2} M \Gamma^{(u,v)} M$$

$$\Rightarrow \boxed{M(\hat{\mu}) = \underbrace{M(0)}_{\text{original } AdS_3 \text{ background}} \left[I + 2\hat{\mu} \ell_s^{-2} \Gamma^{(u,v)} M(0) \right]^{-1}}$$

TsT transformations

What does this correspond to in string theory? Answer:

a TsT transformation

T-duality on u

shift $v \to v - \hat{\mu} u$

T-duality on u

- The TsT transformation is exactly marginal (to one-loop) if

$$\phi \to \tilde{\phi} \,. \quad \sqrt{\tilde{G}}\, e^{-2\tilde{\phi}} = \sqrt{G}\, e^{2\phi} \underbrace{e^{2\phi_0}}_{\text{constant}} \qquad \text{(Buscher's rule)}$$

$\Rightarrow$ TsT is a solution-generating technique of SUGRA

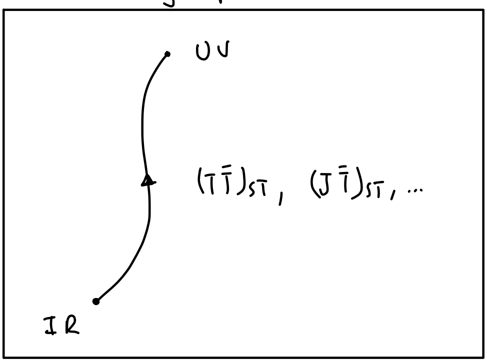
- $M(\hat{\mu})$ satisfies different boundary conditions than $M(0)$ $\Rightarrow$ TsT changes the UV behavior of the dual theory (as expected for $T\bar{T}$)

- TsT works for any background with at least two translational isometries. For $AdS_3 \times S^3 \times T^4$ we have the following interpretation:

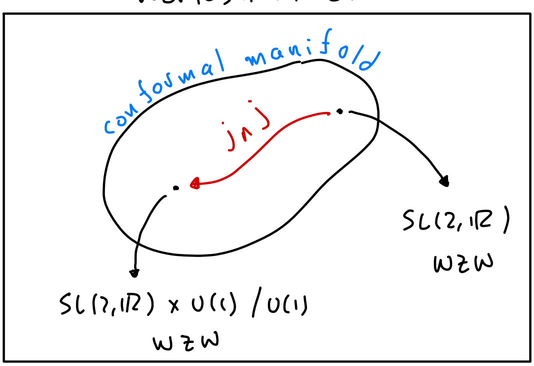
AdS_3			S^3	T^4	deformation
u	v	r	ψ_i	y_i	
x	x				$\hat{\mu}_0 \sum_i T^i \bar{T}^i$
x					$\hat{\mu}_- \sum_i T^i \bar{J}^i$
	x				$\hat{\mu}_+ \sum_i J^i \bar{T}^i$

Summary

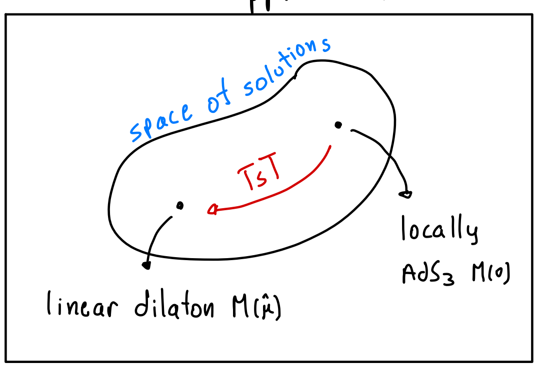
Holographic dual



Worldsheet CFT



SUGRA approximation



TsT black holes

Let us consider the TsT transformation for $T\bar{T}$ (along u and v)

$$BTZ \times S^3 \times T^4 \longrightarrow TsT\ BH \times S^3 \times T^4$$

where the TsT black hole is described by

$$ds_3^2 = \frac{dr^2}{4(r^2 - 4T_u^2 T_v^2)} - \frac{r\,du\,dv - T_u^2 du^2 - T_v^2 dv^2}{1 + \lambda r + \lambda^2 T_u^2 T_v^2}, \qquad \lambda = 2\kappa\hat{\mu}$$

$$B_3 = \frac{r + 2\lambda T_u^2 T_v^2}{2(1 + \lambda r + \lambda^2 T_u^2 T_v^2)}\, du \wedge dv$$

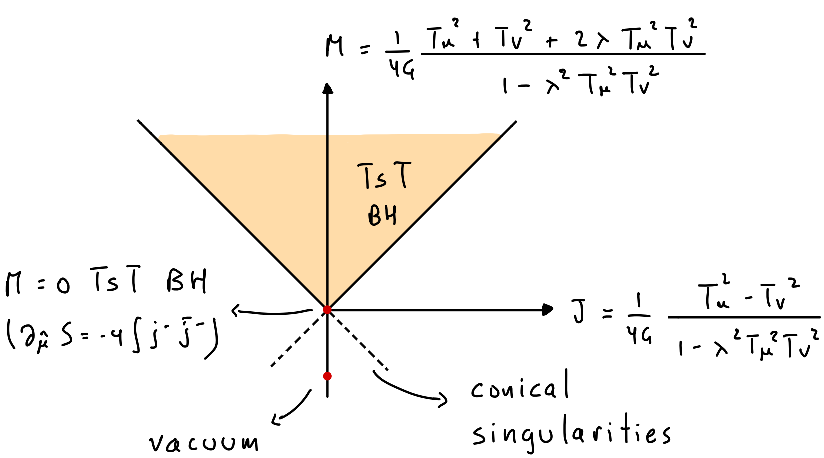
$$e^{2\phi} = \frac{k}{p}\,\frac{1}{1 + \lambda r + \lambda^2 T_u^2 T_v^2} \times \underbrace{(1 - \lambda^2 T_u^2 T_v^2)}_{e^{2\phi_0}}$$

The constant ϕ_0 is fixed by requiring

$$Q_e = \frac{1}{(2\pi l_s)^6}\int_{S^3 \times T^4} e^{-2\phi} * H = p$$

$$Q_m = \frac{1}{(2\pi l_s)^2}\int_{S^3} H = k$$

same (quantized) values as before the deformation

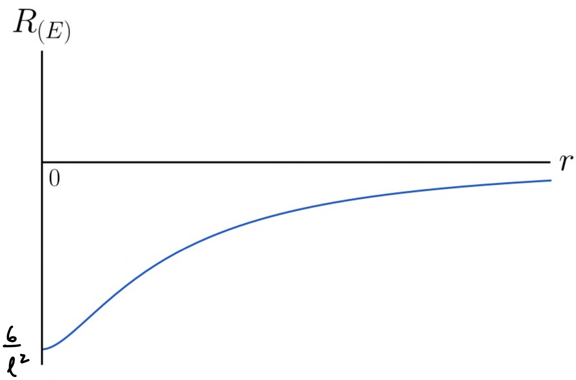


Features:

- Asymptotic behavior as $r \to \infty$:

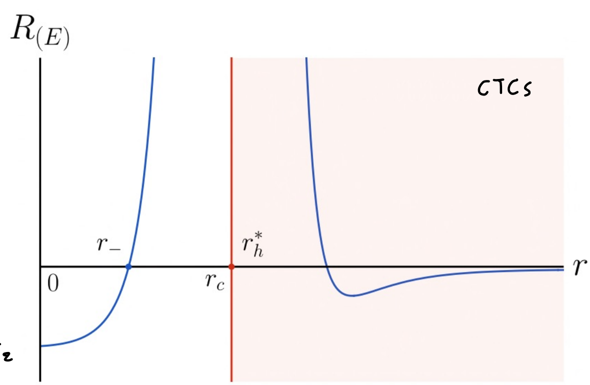
$$ds_3^2 \sim \frac{dr^2}{4r^2} - \frac{1}{\lambda}\, du\, dv, \qquad B \sim \frac{1}{2\lambda}\, du \wedge dv, \qquad \phi \to -\infty$$

$$\underbrace{R_{(E)}}_{\text{Ricci scalar in the Einstein frame}} \propto -\frac{1}{r^2} \to 0$$



IR: $-\frac{6}{\ell^2}$

- Pathologies when $\lambda_0 < 0$: CTCs and a curvature singularity



$-\frac{6}{\ell^2}$

$$r_c = r_h^* = \frac{1 + \lambda^2 T_u^2 T_v^2}{\lambda}$$

Cutoff in double trace $T\bar{T}$:

$$r_{\text{cutoff}} \propto \frac{1}{\lambda}$$

- $\underbrace{\text{Ground state}}_{\text{everywhere smooth geometry}} \to$ analytic continuation of $T_{u,v}$

$$T_u = T_v = \frac{i}{\lambda}\left(1 - \sqrt{1-\lambda}\right)$$

ground state is real iff $\lambda \leq 1$

Exercise: let $T_\mu^2 = T_\nu^2 = -r_0$. Find the equation satisfied by r_0 that guarantees the absence of conical singularities at $r = 2r_0$. Find the solutions to this equation and justify the choice above.

A healthy space of solutions requires:

$$\underbrace{0 \leq \lambda \leq 1}_{\text{compatible with the spectrum of } T\bar{T}\text{-deformed CFTs!}}$$

Recall that the spectrum of M_μ in a $\mathrm{Sym}^p M_\mu$ is given by

$$E(\hat{\mu}) = -\frac{1}{2\hat{\mu}}\left(1 - \sqrt{1 + 4\hat{\mu}\,E(0) + 4\hat{\mu}^2 J(0)^2}\right), \quad J(\hat{\mu}) = J(0)$$

- for large $E(0)$, $E(\hat{\mu})$ becomes complex if $\hat{\mu} < 0$
- for the ground state $E(0) = -\frac{k}{2}$, $E(\hat{\mu}) \in \mathbb{C}$ if $2k\hat{\mu} > 1$

↳ the spectrum is real if $\underbrace{0 \leq \hat{\mu} \leq \frac{1}{2k} \Rightarrow 0 \leq \lambda \leq 1.}_{\text{strong hint the } TsT \text{ solutions are related to single-trace } T\bar{T}}$

Note that $\lambda < 0$ leads to both CTCs and curvature singularities but only the CTCs are associated with complex energy states. This can be seen by turning on additional irrelevant deformations.

The worldsheet spectrum

In the semiclassical limit $k \gg 1$ we can derive the spectrum of the deformed worldsheet theory using spectral flow. Let

$$X^\mu \equiv \text{coordinates after TsT} \qquad \hat{X}^\mu \equiv \text{coordinates before TsT}$$

One can show that

$$\begin{aligned} EOM(X^\mu) &\longrightarrow EOM(\hat{X}^\mu) \\ Vir(X^\mu) &\longrightarrow Vir(\hat{X}^\mu) \end{aligned} \qquad \text{using} \qquad \begin{aligned} \partial\hat{X} &= \partial X - 2\ell_s^{-2}\hat{\mu}\, \partial X \cdot M\, \Gamma \\ \bar{\partial}\hat{X} &= \bar{\partial} X - 2\ell_s^{-2}\hat{\mu}\, \Gamma \cdot M \cdot \bar{\partial} X \end{aligned}$$

The change of coordinates induces twisted boundary conditions on $\hat{X}$:

$$\hat{u}(\sigma + 2\pi) = \hat{u}(\sigma) - 2\pi\gamma^{(u)}, \qquad \hat{v}(\sigma + 2\pi) = \hat{v}(\sigma) - 2\pi\gamma^{(v)}$$

$$\hookrightarrow \gamma^{(\mu)} = \frac{1}{2\pi}\oint (\partial - \bar{\partial})\hat{X} = w + 2\hat{\mu}\,\ell_s^{-2}\frac{1}{2\pi}\oint \left(M_{\alpha v}\,\partial X^\alpha + M_{v\alpha}\,\bar{\partial}X^\alpha\right)$$

$$\gamma^{(u)} = w + 2\hat{\mu}\,E_R$$

$$\gamma^{(v)} = w - 2\hat{\mu}\,E_L.$$

These boundary conditions look like a generalization of winding

$$\Downarrow$$

we can enforce them by a spectral flow transformation:

$$u \to u - \gamma^{(u)} z, \qquad v \to v - \gamma^{(v)}\bar{z}$$

Using the shift of L_0 under spectral flow we obtain:

$$L_0 \to L_0 - \epsilon_L \gamma^{(\mu)} \,, \qquad \bar{L}_0 \to \bar{L}_0 - \epsilon_R \gamma^{(\nu)}$$

Thus, the Virasoro constraints lead to

$$E_L(0) = \epsilon_L(\hat{\mu}) + \frac{2\hat{\mu}}{w} \epsilon_L(\hat{\mu}) \epsilon_R(\hat{\mu})$$

$$E_R(0) = \epsilon_R(\hat{\mu}) + \frac{2\hat{\mu}}{w} \epsilon_L(\hat{\mu}) \epsilon_R(\hat{\mu})$$

$\hookrightarrow$ the spectrum of strings on any TsT transformed background*

matches the spectrum of a $\underbrace{\text{single-trace } T\bar{T}\text{-deformed CFT}}_{\mathrm{Sym}^p \mathcal{M}_\mu}$!

Thermodynamics

The TsT backgrounds feature a horizon at

$r_h = 2 T_u T_v \longrightarrow$ same as before the TsT transformation

Since the low energy effective theory is just SUGRA the entropy is

$$S = \frac{A}{4G} = \frac{\pi}{4G} \frac{(T_u + T_v)}{1 - \lambda T_u T_v}$$

$$= 2\pi \left(\sqrt{\underbrace{kp}_{\frac{c}{6}}\, E_L \left(1 + \underbrace{\frac{2\lambda}{kp}}_{\frac{2\hat{\mu}}{p}} E_R\right)} + \sqrt{kp\, E_R \left(1 + \frac{2\lambda}{kp} E_L\right)} \right)$$

$\hookrightarrow$ matches the entropy in the single trace case!

$$T_L = \frac{1}{\pi} \frac{T_u}{1 + \lambda T_u T_v}$$

$\hookrightarrow$ consistent with the first law $\delta S = \frac{1}{T_L} \delta E_L + \frac{1}{T_R} \delta E_R$

Comments and conclusions

- The matching of the spectrum is consistent with the fact that the long string sector is captured by $\mathrm{Sym}^p \mathcal{M}$.
- The entropy also matches the $\mathrm{Sym}^p \mathcal{M}_\mu$ formula.

  The $\mathrm{Sym}^p \mathcal{M}_\mu$ derivation relied only on

  (1) modular invariance

  (2) energies of the vacuum

  Tentative explanation: the marginal deformation Φ of $\mathrm{Sym}^p T^4$ must preserve (1) + (2)!
- Additional evidence for this from the gravitational charges of the ground state geometry:

$$E = \frac{k p}{4\lambda}\left(\sqrt{1-\lambda} - 1\right) = \underbrace{\frac{p}{2\hat{\mu}}\left(\sqrt{1 - \frac{\hat{\mu} c}{3p}} - 1\right)}_{\substack{\text{same energy of } \mathrm{Sym}^p \mathcal{M}_\mu \\ \text{used in derivation of } S}}$$

- We have several pieces of evidence that TsT transformations are related to the single-trace $T\bar{T}$-deformation.
- What happens at $k=1$? Can we prove this correspondence exactly, to the same level as string theory on AdS_3?