

Problem set 1 (due October 11)

1. (2pts) Consider a system of particles with positions $\vec{r}_i, i = 1, \dots, N$. Let $\vec{r}'_i$ denote the position of the i^{th} particle relative to the center of mass $\vec{R}$ such that

$$\vec{r}_i = \vec{r}'_i + \vec{R}$$

In the $\vec{r}'$ coordinates, it's easy to show that the total momentum vanishes, i.e. $\vec{P}' = \sum_i \vec{p}'_i = 0$, which implies that $\dot{\vec{P}}' = 0$. Applying Newton's second law would tell us that there are no external forces acting on the system, even though we didn't make any assumptions on the external forces. Explain why.

2. Consider a general transformation between an inertial frame O and a frame O' (not necessarily inertial) such that the position $\vec{r}'$ of a point particle in O' is given with respect to the position $\vec{r}$ in O by

$$\vec{x}' = \vec{f}(\vec{x}, t)$$

- (a) (2pts) Find the conditions $\vec{f}(\vec{x}, t)$ must satisfy such that Newton's second law holds in O' .
 - (b) (2pts) How are these conditions related to Galilean transformations?
 - (c) (2pts) Write down the transformation rule of a generic element in the Galilean group and derive the composition rule for two Galilean transformations. Use this to show that the Galilean group satisfies all the axioms of a group.
 - (d) (2pts) Find how the Lagrangian of a free point particle (i.e. a particle without any forces acting on it) transforms under Galilean transformations. Show that the transformed Lagrangian is equivalent to the original one so the Euler-Lagrange equations are invariant under Galilean transformations.
3. (4pts) Consider a point particle with mass m on the surface of the Earth at position $\vec{r} = (x, y, z)$. The Earth is assumed to be a sphere of radius R that rotates with angular velocity w around the z axis. Let $\vec{w} = (0, 0, w)$ denote the angular velocity vector. Find the Lagrangian and use the Euler-Lagrange equations to identify the spurious forces. Note that w is not assumed to be constant, so there will be an additional force from those discussed in class. Show the direction in which this force points.
 4. Use the Euler-Lagrange equations to obtain the following:
 - (a) (2pts) The Lorentz force on a single point particle. In this case the Lagrangian is given by $L(x^i, \dot{x}^i, t) = T(\dot{x}^i) - W(x^i, \dot{x}^i)$ where the generalized potential reads

$$W(x^i, \dot{x}^i) = A_k(x^i, t)\dot{x}^k + \Phi(x^i, t)$$

Express the final answer in terms of the electric $\vec{E}$ and magnetic $\vec{B}$ fields, which are vector fields whose components are given by

$$E_i = \frac{\partial A_i}{\partial t} - \frac{\partial \Phi}{\partial x^i}, \quad B_i = \frac{1}{2} \epsilon_{ijk} \left(\frac{\partial A_j}{\partial x^k} - \frac{\partial A_k}{\partial x^j} \right)$$

(b) (2pts) The geodesic equation of a free particle in a Riemannian manifold with metric $g_{ij}(x^k) = g_{ji}(x^k)$. The geodesic equation can be derived from the following action

$$S = \frac{1}{2} m \int g_{ij} \dot{x}^i \dot{x}^j d\lambda, \quad \text{where} \quad \dot{x}^i = \frac{dx^i}{d\lambda}$$

Express the answer in terms of the Christoffel symbol

$$\Gamma_{ij}^k = \frac{1}{2} g^{km} (\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij}), \quad \partial_i = \frac{\partial}{\partial x^i}$$

(c) (Bonus, 2pts) Recall that the metric of the configuration manifold C is defined to be $m_{ij} = \frac{\partial^2 L}{\partial \dot{x}^i \partial \dot{x}^j}$. In the absence of any forces (i.e. no potential), what can you deduce about the curve traced by the particle in C ?

5. (2pts) Consider a particle in $\mathbb{R}^2$ with coordinates (x, y) subject to a conservative force with potential $V(y)$, i.e. a force that only acts in the y direction. We are interested in the curve $C = \{y = y(x)\}$ for which the time required to go from a point p_1 to a point p_2 is minimal. Write down a functional for this problem (minimization of time, not of the action) in terms of $V(y)$ and compute the corresponding Euler-Lagrange equations.