Problem set 2 (due October 27)

1. Consider a point particle lying a distance $h \ll R$ from the surface of the Earth, the latter of which is assumed to be a sphere of radius R rotating with constant angular velocity w. The particle is subject to the force of gravity and starts at rest at a point $p_0 = (r_0, \varphi_0, z_0) = (R + h, 0, 0)$ in cylindrical coordinates, i.e. it starts a distance h above the equator with velocity $\vec{v} = 0$. With this assumption, the particle is confined to move on the $z = 0$ plane.

   (a) (1pt) How many degrees of freedom are needed to describe the motion of the particle?

   (b) (2pt) Find the Lagrangian, identify the cyclic coordinates, and the corresponding conserved quantities

   (c) (2pt) Using the initial conditions and conservation laws, write down the Euler-Lagrange equations in terms of w and r_0

   (d) (2pt) Assuming that $\left(\frac{wr_0^2}{r^2}\right)^2 \ll \frac{g}{r}$ solve the equations of motion and find how much the particle deviates from φ_0 when it reaches the ground

2. In class, we showed that for every continuous symmetry of the Lagrangian there exists a conserved quantity. Consider now the more general family of maps

$$q_i(t) \mapsto Q_i(s,t), \quad t \mapsto T(s,t) \qquad \text{where} \qquad Q_i(0,t) = q_i(t), \quad T(0,t) = t \tag{1}$$

   where $s \in \mathbb{R}$ and

$$\left.\frac{\partial^2 T}{\partial s \partial t}\right|_{s=0} = \left.\frac{d}{dt}\frac{\partial T}{\partial s}\right|_{s=0} \tag{2}$$

   This family of maps is a continuous symmetry of the Lagrangian whenever the following one-form is independent of s,

$$L(Q_i(s,t), \dot{Q}_i(s,t), T)\, dT(s,t) + \frac{d}{dT} F(Q_i(s,t), T(s,t), s)\, dT(s,t) \tag{3}$$

   where $F(Q_i(s,t), T(s,t), 0) = 0$. In particular, note that

$$\dot{Q} = \frac{dQ}{dT} = \frac{\partial t}{\partial T}\left(\frac{\partial Q_i}{\partial q_j}\frac{\partial q_j}{\partial t} + \frac{\partial Q_i}{\partial t}\right) \tag{4}$$

   (a) (3pts) Find the conserved quantity $J(q_i, \dot{q}_i, t)$ associated with this symmetry (hints: (1) you can extract J from an expression of the form $(\frac{d}{dt}J)dt = 0$; (2) you will need to show and use $\frac{\partial L}{\partial t} = -\frac{d}{dt}\left(\dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L\right)$)

   (b) (2pts) Show that the following action is invariant under all time reparametrizations $t \mapsto T(t)$

$$S = \int \sqrt{\dot{\vec{r}} \cdot \dot{\vec{r}}}\, dt \tag{5}$$

Letting $T = t + s_n t^n$, where s_n is a constant, compute the conserved quantity associated with each s_n.

Bonus (1pt): what's the interpretation of this result?

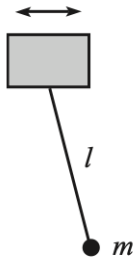
(c) (2pts) Consider a system of particles with Lagrangian

$$L = \frac{1}{2} m_i |\dot{\vec{r}}_i|^2 - V(r_i - r_j) \tag{6}$$

Idenfity the function F and compute the conserved current associated with a (Galilean) boost $\vec{r}_i \to \vec{r}_i + \vec{v}t$. This completes the derivation of conserved quantities associated with Galilean transformations discussed in class.

Bonus (1pt): what's the physical interpretation to this result?

3. Consider a pendulum that consists of a point particle of mass m hanging from a block of mass M by a massless rod of length l. The block moves in the horizontal direction under the influence of a potential $V(x)$, as illustrated below



(a) (1pts) How many degrees of freedom are needed to describe the evolution of the system?

(b) (2pts) Write down the Lagrangian and the corresponding equations of motion

(c) (2pts) Let $V(x) = kx^2$ and obtain an ordinary second-order differential equation for the motion of m

(d) (1pt) Solve the equations of motion in the small angle approximation.

Bonus (1pt): find the choice of parameters for which the answer diverges and the small angle approximation breaks down. What's the physical interpreation of this choice?