

Problem set 3 (due December 6)

1. (2pts) Consider a particle in three dimensions where two components of the angular momentum, L_i and L_j , and one component of the momentum p_k (with $i \neq j \neq k$) are conserved. Use the properties of the Poisson bracket to find the components of $\vec{L}$ and $\vec{p}$ that are also conserved.
2. Consider a particle of mass m moving in one dimension under a constant acceleration a .
 - (a) (1pt) Write down the Lagrangian and the corresponding Hamiltonian.
 - (b) (1pt) Find a canonical transformation that makes the position of the particle a cyclic coordinate.
 - (c) (2pt) Solve Hamilton's equations and show that they reproduce the solution to the Euler-Lagrange equations.
3. (1pt) Consider a generating function of the second kind $F_3(p, Q, t)$, i.e. a generating function that depends on the old momenta $\{p_1, \dots, p_n\}$ and the new coordinates $\{Q_1, \dots, Q_n\}$. Use the variational principle to derive the expressions for the old coordinates q_i , the new momenta P_i , and the new Hamiltonian we discussed in class.
4. Consider the following transformation $(q, p) \mapsto (Q, P)$

$$Q = q \cos \alpha - p \sin \alpha, \quad P = q \sin \alpha + p \cos \alpha, \quad \alpha \in \mathbb{R} \quad (2)$$

- (a) (1pts) Verify that this transformation is canonical for any value of α
 - (b) (2pts) Write a generating functional for the canonical transformation.
 - (c) (2pts) The generating functional will become singular at either $\alpha = 0$ or $\alpha = \pi/2$. Propose another generating functional that is valid at the value where the generating functional of part (b) is singular.
5. Consider the following action (recall that repeated indices imply summation, e.g. $q_i^2 = \sum_i (q_i)^2$)

$$S[q_i] = \int \left(\frac{1}{2} m \dot{q}_i^2 - \frac{k}{q_i^2} \right) dt, \quad i = 1, \dots, n \quad (3)$$

The Lagrangian is invariant under scalings

$$(q_i, t) \mapsto (Q_i, T) = (e^\alpha q_i, e^{2\alpha} t) \quad (4)$$

and the conserved quantity reads

$$J = m q_i \dot{q}_i - t \left(m \dot{q}_i^2 + \frac{2k}{q_i^2} \right) \quad (5)$$

- (a) (2pts) Write the Hamiltonian H for this system and express the conserved quantity in terms of the canonical variables.
- (b) (2pt) Verify that J is a symmetry generator in the Hamiltonian formalism

(c) (2pt) Use J to find the infinitesimal transformation of the canonical coordinates (to linear order in α)

(d) (2pt) Compute the infinitesimal change in the Hamiltonian of part (a) and interpret the result.

(e) (Bonus: 3pts) Compute the infinitesimal transformation of the coordinate q_i to second order in α . From this expression deduce the full transformation of q_i , i.e. make an ansatz for $Q_i(q_i, \alpha)$, and verify that it's compatible with the expected scaling transformation.