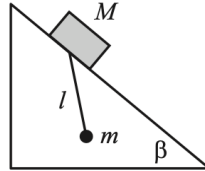
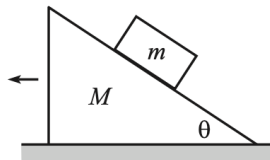


Additional problems

1. Consider a block of mass M sliding down an inclined plane at an angle β . A pendulum of length l and mass m hangs from M as shown in the figure.



- (a) Write down the Lagrangian
 - (b) Find the equations of motion
 - (c) For small oscillations, find the normal modes and their frequencies
2. Consider a block of mass m sliding down an inclined plane of mass M at an angle θ . The plane lies on a frictionless surface so it's able to move as shown in the figure.



- (a) Write down the Lagrangian
 - (b) Find the equations of motion
 - (c) What is the acceleration of the plane?
3. A particle moves in periodic motion in one dimension under the influence of a potential $V(x) = k|x|$ where k is a constant. Using action-angle variables, find the period of the motion as a function of the particle's energy.
 4. A particle of mass m moves in one dimension under the influence of a potential $V(x) = -k/|x|$ where k is a constant. For energies that are negative, the motion is bounded and oscillatory. In this case, use action-angle variables to find an expression for the period of motion as a function of the particle's energy.
 5. Show directly that the following transformation is canonical for any value of α

$$Q = \tan^{-1} \frac{\alpha q}{p}, \quad P = \frac{\alpha q^2}{2} \left(1 + \frac{p^2}{\alpha^2 q^2} \right) \quad (1)$$

6. Show directly that the following transformation is canonical and find a generating function

$$Q_1 = q_1, \quad P_1 = p_1 - 2p_2, \quad Q_2 = p_2, \quad P_2 = -2q_1 - q_2 \quad (2)$$

7. Consider the simple harmonic oscillator and the following function in phase space

$$J(q, p, t) = (p + im\omega q)e^{-i\omega t} \quad (3)$$

(a) Show by means of Poisson brackets that J is a constant of motion

(b) Note that J^* is also a constant of motion. How are J and J^* are related to the Hamiltonian?

(c) Using the solution to the simple harmonic oscillator, give a physical interpretation to these constants of motion.

8. A particle with mass m , position $\vec{r}$ and momentum $\vec{p}$ has angular momentum $\vec{L} = \vec{r} \times \vec{p}$.

(a) Evaluate $\{r_j, L_k\}$, $\{p_j, L_k\}$, $\{L_j, L_k\}$ and $\{L_i, \vec{L}^2\}$.

(b) The Runge-Lenz vector is defined as $\vec{A} = \vec{p} \times \vec{L} - \hat{r}$ where $\hat{r} = \vec{r}/|\vec{r}|$. Show that $\{L_a, A_b\} = \epsilon_{abc} A_c$.

(c) If the system is described by the Hamiltonian $H = (p^2/2m) - 1/r$, show that $\vec{A}$ is conserved using Poisson brackets.

9. Let us consider the differentiable manifold $M = \mathbb{R}^3$ with Cartesian coordinate system (x^1, x^2, x^3) . In three-dimensions the dimensionality of the space of 1-forms and 2-forms equals 3. Hence, it's not surprising that in this case every vector field $\vec{A}$ corresponds to a 1-form w_A^1 and a 2-form w_A^2 defined by the conditions

$$w_A^1(\vec{\xi}) = \vec{A} \cdot \vec{\xi}, \quad w_A^2(\vec{\xi}, \vec{\eta}) = \vec{A} \cdot (\vec{\xi} \times \vec{\eta}) \quad (4)$$

In the basis $e_i = \frac{\partial}{\partial x^i}$ and $e^{*i} = dx^i$,

- Write the components of the 1-form w_A^1 and the 2-form w_A^2 in terms of the components of the vector field $\vec{A}$.
- Find the relationship of the divergence and curl of a vector field $\vec{A}$ to the exterior derivatives of the 1-form and 2-form defined above

10. Using the results above show that $\vec{\nabla} \times \vec{\nabla} f = 0$ and $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{A}) = 0$ are both consequences of $d^2 = 0$.